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PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

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PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

By

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PREFACE

Projective Differential Geometry is largely a product of the first three decades of the twentieth century. The theory has been developed in five or more different languages, by three or four well-recognized methods, in various and sundry notations, and has been published partly in journals not readily accessible to all.

The author's aim in writing this book was to organize an exposition of these researches. He desired to coordinate the results achieved on both sides of the Atlantic so as to make the work of the European geometers more readily accessible to American students, and so as to make better known to others the accomplishments of the American school. The author has made use of those of his own results which have been published in journals, and has also included occasional new results hitherto unpublished.

Since this book was not designed to be an exhaustive treatise no attempt was made to include in it all existing projective differential geometry. Certainly, topics already adequately treated in other books could be somewhat neglected in this one. So, for example, periodic sequences of Laplace and the theorems of permutability receive here only passing mention because these subjects are extensively discussed in books by Tzitzéica and Eisenhart. Moreover, certain things could be neglected because they seemed to be primarily analytical rather than geometrical in their nature; by way of illustration may be cited the calculation of complete systems of invariants and covariants.

As to arrangement of material, it is hoped that the order in which topics spontaneously occurred to the author may prove to be the natural one. There is no simpler theory to begin with than that of curves, to which Chapter I is devoted. The theory of ruled surfaces, which occupies Chapter II, is the next simplest. The elements of both of these theories are prerequisite for the study of surfaces in ordinary space, which is found in Chapter III. The subject of conjugate nets, as developed in Chapter IV, leads easily to transformations of surfaces in Chapter V. In Chapter VI some parts of these projective considerations are specialized so as to show their connections with metric and affine geometry. In Chapter VII the projective theory of surfaces in hyperspace is amplified to some extent and is generalized in order to introduce varieties of more dimensions than two. Finally, Chapter VIII contains a number of miscellaneous topics which it seemed unwise to exclude altogether and which are to be regarded as more or less supplementary.

Certain mathematical attainments on the part of the reader are prerequisite to understanding this book. Some previous acquaintance with the fundamentals of analytic projective geometry is highly desirable, as familiarity with homogeneous coordinates is assumed from the outset. The reader should be acquainted with, or have constantly at hand, such a book as Graustein's *Introduction to Higher Geometry*. Moreover, the reader is supposed on occasion to have some knowledge of differential equations, power series, and other portions of analysis and algebra.

There is a list of exercises at the end of each chapter. These are designed to give the reader practice in actually working at problems in geometry. Many of them are also intended to point the way to further extensions of the theory that may be found in the literature. Some of them contain results not previously published.

No attempt has been made to prepare a complete bibliography. However, there is a working bibliography at the end of the text, in which the references are of two kinds. Some are to the original memoirs. Others are to the literature thought to be most convenient for the reader. It is believed that the latter references will not be misleading to a student who wishes to follow a subject to its source, but will usually serve to guide him eventually to the original publication.

The author hereby gratefully renders due homage to all the great company of geometers living and dead whose published researches have been drawn upon for the enrichment of these pages. Having availed himself freely of the labors of others, he wishes that it were practicable in *every* instance to assign credit to whom credit is due for original discovery. The perfection of this difficult task must be left to the historian of Projective Differential Geometry, but most especially the author wishes to avoid the appearance of claiming credit to himself for any discovery made first by another.

It is a particular pleasure for the author to acknowledge his debt to Wilczynski for early instruction in Projective Differential Geometry, and for the first suggestion in 1918 that this book should be written. The author is also happy to thank the Italian geometers Fubini, Bompiani, Terracini, Castelnuova, and others for their cordiality and many courtesies extended to him while he was studying geometry in Italy as Guggenheim Fellow. To his colleague, Professor Bliss, the author is indebted for constant encouragement, for material assistance, and for wise counsel on many occasions. To the incisive and penetrating criticisms of Dr. Mendel are due many improvements in the manuscript.

Ernest P. Lane

University of Chicago
April, 1931

CONTENTS

CHAPTER I

CURVES

Introduction. Before beginning to develop systematically the theory of projective differential geometry with which this book is concerned it seems appropriate to describe here in a general way the nature of this kind of geometry, and to comment briefly on the contents of the first chapter.

It is well known that any plane has the property that the straight line joining any two distinct points in it lies entirely in the plane. In general, a space of any number n of dimensions which has this property is called a *linear space* and is denoted hereinafter by S_n. In such a space a non-singular *projective transformation* can be defined analytically as the linear homogeneous substitution that is represented in homogeneous point coordinates by a system of equations of the form

$$\rho y_i = \sum_{j=1}^{n+1} a_{ij} x_j \qquad (i=1, \ldots, n+1;\ a_{ij} = \text{const.}) ,$$

where the determinant of the coefficients a_{ij} is different from zero, and ρ is an arbitrary factor of proportionality not zero. There are $(n+1)^2$ coefficients a_{ij} of which $n(n+2)$ are independent.

The *projective geometry* of a configuration is the theory of those properties of the configuration that remain invariant under all projective transformations of the space in which the configuration lies. Some examples of projective invariants are the straightness of a line, the relation of united position of point and line, and the cross ratio of four points on a line. These cannot be changed by a projective transformation. Projective geometry may be contrasted with *metric geometry* which studies properties that remain unchanged by rigid motions. For example, the distance between two points and the angle between two lines are metric invariants but are not projective invariants.

The *differential geometry* of a configuration is the theory of the properties of the configuration in the neighborhood of a general one of its elements. In particular, the differential geometry of a curve is concerned with the properties of the curve in the neighborhood of a general one of its points. In analytic geometry the tangent line at a point of a curve is customarily defined to be the limit of the secant line through this point and a neighboring point on the curve as the second point approaches the first along the

curve. This definition of the tangent at a point of a curve illustrates the nature of differential geometry in that it requires a knowledge of the curve only in the neighborhood of the point and involves a limiting process. These features of differential, or infinitesimal, geometry show why this geometry employs so extensively the differential calculus. Differential geometry may be contrasted with *integral geometry* which views a configuration as a whole. The problem of finding the number of intersections of a straight line and a conic in the same plane is a problem of integral geometry, since its solution requires a knowledge of the entire line and conic.

These preliminary remarks enable us to describe quite accurately the nature of projective differential geometry in a few words. *Projective differential geometry* is, as the name implies, the theory of the projectively invariant differential properties of geometrical figures. This book is mainly devoted to the projective differential geometry of curves and surfaces. Such other configurations as may appear play a subordinate rôle.

The first chapter is devoted to the projective differential geometry of curves. In the first three sections analytic curves are considered in a linear space of n dimensions. The foundations are laid for Wilczynski's theory of curves, which is based upon a consideration of the invariants and covariants of an ordinary linear homogeneous differential equation under a suitably chosen group of transformations. We do not elaborate this theory here, however, even to the extent of computing a complete system of invariants. In the remaining four sections we adopt rather the point of view of Halphen who studied the projective differential geometry of curves, first in the plane in 1878 and then in ordinary space in 1880, by means of power series expansions in non-homogeneous coordinates, the coefficients in the expansion at a point of a curve being of course constants when the point is fixed, and being absolute projective differential invariants of the curve when the point is allowed to vary along the curve. This method has the advantage of being direct and simple and is easily coordinated with Wilczynski's method, as Wilczynski himself has shown. The ordinary osculants of curves are introduced, and canonical forms are obtained for the power series expansions used. The coordinate systems which lead to these expansions are completely characterized geometrically.

1. Definition and space of immersion of a curve. After the introduction of certain notations and conventions, an analytic curve will be defined in this section. Then analytic conditions necessary and sufficient that the locus just defined may reduce to a fixed point, to a straight line, or to a plane curve will be deduced. The idea of the immersion of a curve in a linear space will be introduced and the *space of immersion* of a configuration will be de-

fined and illustrated. The final result of this section will be the formulation of conditions necessary and sufficient that a curve in a linear space S_n may be immersed in a linear subspace S_k of S_n.

In a linear space of n dimensions S_n let us consider a point P with $n+1$ projective homogeneous coordinates $x_1, \ldots, x_{n+1}$. It is often convenient to speak of the point P as *the point* x, and to indicate this point by P_x. Sometimes the set $(x_1, \ldots, x_{n+1})$ is thought of as a *vector* x, and then each of the coordinates $x_1, \ldots, x_{n+1}$ is a *component* of this vector. A *scalar* may be regarded as a vector with only one component.

A *curve* can be described qualitatively as a one-parameter family, or single infinity, of points. More precisely, *an analytic* curve* can be defined as follows. *If the coordinates x of a point P are single-valued analytic functions of one independent variable t, then the locus of the point P, as t varies over its range, is an analytic curve C.* Analytic curves are the only curves that will be considered in this book, although ordinarily it would be sufficient to suppose that the functions involved possess a certain number of derivatives. When the coordinates x of the point P are expressed as functions of t by equations of the form

$$x_i = x_i(t) \qquad (i=1, \ldots, n+1) , \tag{1}$$

these are spoken of as *the parametric equations* of the curve C. If we think of x as a general one of the coordinates x_i, the subscript i can be dropped, and so equations (1) can be replaced by what is called *the parametric vector equation* of the curve C, namely,

$$x = x(t) . \tag{2}$$

There are various possibilities as to the nature of the locus C just defined. First of all, it may reduce to *a single fixed point*, being, in this case only, not a proper curve. In fact, if the point P remains fixed as t varies, the ratios of the coordinates x are constants, and hence there exists a scalar function λ of t, which is not zero and is such that

$$\lambda x = a \qquad (x,\ a \text{ vectors; } a = \text{const.}) .$$

Therefore x satisfies the *ordinary linear homogeneous differential equation of the first order*

$$x' + px = 0 \qquad (p = \lambda'/\lambda) , \tag{3}$$

* Goursat-Hedrick, 1904. 1, p. 407. References are to the Bibliography at the end of the text, unless otherwise indicated. The figure following the year indicates, in each instance, the order in the list appearing under that year in the Bibliography.

accents indicating differentiation with respect to t. Conversely, let us suppose that the coordinates x satisfy an equation of the form (3). The general solution of this equation is

$$x = ae^{-\int p\,dt} \qquad (a = \text{const.})\ ;$$

and $n+1$ particular solutions are obtained by giving $n+1$ values to the arbitrary constant a while the exponential function remains the same. Therefore the ratios of the coordinates x are constants; so the point P is fixed and the locus C reduces to a single fixed point. Consequently we have the theorem:

A necessary and sufficient condition that a point be fixed, when the coordinates of the point are functions of a single variable, is that these coordinates satisfy an ordinary linear homogeneous differential equation of the first order.

A second possibility as to the nature of the locus C will now be considered. If the locus C does not reduce to a single fixed point but is *a straight line*, let P_a, P_b be two distinct fixed points on this line, while P_x is a variable point on it. Then it is possible to express x in the form

$$x = \lambda a + \mu b\ ,$$

where λ, μ are linearly independent scalar functions of t, and a, b are vector constants. Differentiating twice and eliminating a, b by use of a determinant we find that x is a solution of the differential equation

$$\begin{vmatrix} x & \lambda & \mu \\ x' & \lambda' & \mu' \\ x'' & \lambda'' & \mu'' \end{vmatrix} = 0\ ,$$

which, by expanding according to the elements in the first column, can be written in the form

$$x'' + 2p_1 x' + p_2 x = 0\ , \tag{4}$$

because the coefficient of x'' is not zero. Conversely, let us suppose that x satisfies an equation of the form (4), but satisfies no equation of the form (3), so that the point P_x is not fixed. The general solution of the equation (4) is a linear combination of two particular linearly independent solutions λ, μ with constant coefficients a, b; and $n+1$ particular solutions are obtained by giving $n+1$ pairs of values to the arbitrary constants a, b. Therefore the locus C is the straight line joining the two fixed points P_a, P_b

whose coordinates are, respectively, the $n+1$ values of the two coefficients a, b. Hence follows the theorem:

A curve (2) *is not a fixed point but is a straight line in case the coordinates* $x(t)$ *of a variable point on the curve satisfy an equation of the form* (4) *but satisfy no equation of the form* (3).

Similarly it can be shown that *a curve* (2) *is not a fixed point nor a straight line but is a plane curve if, and only if,* x *satisfies an equation of the form*

$$x'''+3p_1x''+3p_2x'+p_3x=0 , \tag{5}$$

the coefficients p_1, p_2, p_3 *being scalar functions of* t, *but does not satisfy an equation of the form* (4) *nor one of the form* (3).

In general, a curve C in a linear space S_n is said to be *immersed* in a linear subspace S_k ($k \lesseqgtr n$) of S_n in case C is in S_k but is not in a linear subspace S_h of S_k with $h<k$. If a curve C is immersed in a linear space S_k then S_k may be called the *space of immersion* of C. In this sense *the space of immersion of a configuration is the linear space of least dimensions that contains it.* For example, the space of immersion of a proper conic in ordinary space is the plane in which the conic lies.

If a curve C is immersed in a linear subspace S_k of a space S_n then it is possible to express the coordinates x of a variable point P_x on C in the form

$$x=\sum_{i=1}^{k+1}\lambda_i a^{(i)} ,$$

where $\lambda_1, \ldots , \lambda_{k+1}$ are linearly independent scalar functions of t, and $a^{(1)}$, $\ldots$, $a^{(k+1)}$ are vector constants each of which has for components the coordinates of a fixed point in the space S_k. Differentiating $k+1$ times and eliminating $a^{(1)}, \ldots , a^{(k+1)}$, we reach the following conclusion:

A curve (2) *in a linear space* S_n *is immersed in a linear subspace* $S_k (k \lesseqgtr n)$ *of* S_n *if, and only if, the coordinates* $x(t)$ *of a variable point on the curve satisfy a linear homogeneous differential equation of order* $k+1$ *and do not satisfy such an equation of order less than* $k+1$.

2. Transformations. Invariants and covariants. The contents of this section may be summarized as follows. When the curve C, defined by the parametric equations (1), is immersed in the linear space S_n, an easy calculation leads to an ordinary linear homogeneous differential equation of order $n+1$ satisfied by the coordinates $x(t)$ of a variable point on C. This equation is called *a differential equation of the curve C;* and conversely, C is called *an integral curve of the equation.* When the equation is given, the

curve is determined except for a projective transformation; so the equation does not define the curve uniquely. Moreover, when the curve is given, the equation is not determined uniquely, since an arbitrary transformation of the proportionality factor of the homogeneous coordinates, and an arbitrary transformation of the parameter that varies along the curve, can still be made without disturbing the integral curves. These considerations lead to the definitions of invariants and covariants, which are fundamental in Wilczynski's method of studying curves.

Let us consider a curve C with the parametric equations (1), and let us suppose that C is immersed in a space S_n. The wronskian* of the coordinates x can be written in the form

$$(x,\ x',\ .\ .\ .\ ,\ x^{(n)})$$

by writing within parentheses only a typical row of the determinant. This wronskian does not vanish identically. For, if it did, the coordinates x would be linearly dependent, and the curve C would be in a hyperplane S_{n-1}; this conclusion is contrary to the assumption that the curve C is immersed in the space S_n. Therefore, if the coordinates x are substituted one at a time in the differential equation

$$x^{(n+1)}+(n+1)p_1x^{(n)}+\ .\ .\ .\ .\ +p_{n+1}x=0\ , \tag{6}$$

the resulting $n+1$ linear algebraic equations can be solved uniquely for the coefficients p. The equation (6) with its coefficients thus determined is said to be *a differential equation of the curve C;* and conversely, C is called *an integral curve of* (6).

When the differential equation (6) is given, any $n+1$ linearly independent solutions $x_1, \ .\ .\ .\ , x_{n+1}$ of it can be used as homogeneous coordinates of a point P_x which, as t varies, generates an integral curve C_x of equation (6). Let us consider the effect of a non-singular projective transformation,

$$y_i=\sum_{j=1}^{n+1}a_{ij}x_j \qquad (i=1,\ .\ .\ .\ ,\ n+1;\ a_{ij}=\text{const.})\ ,$$

on the curve C_x. This transformation associates with the variable point P_x a variable point P_y which, as t varies, generates a curve C_y which is also an integral curve of equation (6), since the coordinates $y_1, \ .\ .\ .\ , y_{n+1}$ may be shown to satisfy (6). The theory of linear differential equations teaches us that the most general integral curve of equation (6) can be thus obtained as

* Goursat-Hedrick, 1917. 1, p. 103.

a projective transform of any particular one. The general integral curve, just as the general projective transformation, depends on $n(n+2)$ parameters. Hence we reach the conclusion:

In a space S_n the $\infty^{n(n+2)}$ integral curves of a given differential equation (6) *are all projectively equivalent. A differential equation* (6) *defines a curve in the space S_n except for a projective transformation; and a geometric theory based on the equation must be a projective theory.*

The effect of *the transformation of proportionality factor,*

$$x=\lambda(t)\xi \qquad (\lambda \text{ scalar} \neq 0)\,, \tag{7}$$

on equation (6) is found by calculating the successive derivatives of x up to and including the one of order $n+1$ and then substituting them in (6) and collecting the coefficients of the various derivatives of ξ. The result is a differential equation for ξ, of the same form as equation (6). The coefficients $\pi_1, \ldots, \pi_{n+1}$ of the new equation are found to be given by the formulas

$$\begin{cases} \lambda\pi_1 = \lambda'+p_1\lambda\,, \\ \lambda\pi_2 = \lambda''+2p_1\lambda'+p_2\lambda\,, \\ \ldots \quad \ldots \\ \lambda\pi_{n+1}=\lambda^{(n+1)}+(n+1)p_1\lambda^{(n)}+\cdots+p_{n+1}\lambda\,. \end{cases} \tag{8}$$

Since x and ξ differ only by a scalar factor of proportionality, the points P_x and P_ξ coincide. Therefore we have the theorem:

Equation (6) *and any equation into which it can be transformed by a transformation of proportionality factor* (7) *have the same integral curves.*

It is possible to choose the proportionality factor λ so as to simplify the differential equation. If, for instance, the function λ is a solution of equation (6) then $\pi_{n+1}=0$. In particular, we may choose $\lambda=x_1$; this choice of λ amounts to introducing non-homogeneous coordinates, $1, \xi_2, \ldots, \xi_{n+1}$ by placing

$$\xi_i=x_i/x_1 \qquad (i=1, \ldots, n+1)\,.$$

However, if the function λ is a solution of the equation

$$\lambda'+p_1\lambda=0\,,$$

then $\pi_1=0$, and the remaining coefficients are given* by the formula

$$P_k=e^{\int p_1 dt}\sum_{i=0}^{k} C_{k,i}\, p_i \frac{d^{k-i}}{dt^{k-i}} e^{-\int p_1 dt} \qquad (k=2, 3, \ldots, n+1)\,,$$

* Wilczynski, 1906. 1, p. 16.

where it is understood that $p_0=1$. In particular, we have

$$(9) \qquad P_2=p_2-p_1^2-p_1' , \qquad P_3=p_3-3p_1p_2+2p_1^3-p_1'' .$$

These functions P are called *seminvariants*, because they are invariants under, or are unchanged by, the transformation (7) but there is still another transformation to be considered. That *the functions P are in fact absolute invariants under the transformation* (7) can readily be verified by means of the formulas (8); for instance, an easy calculation shows that

$$\pi_2-\pi_1^2-\pi_1'=p_2-p_1^2-p_1' .$$

The effect of *the transformation of parameter,*

$$(10) \qquad u=u(t) \qquad (u'\neq 0) ,$$

on equation (6) is found by changing the independent variable in (6) from t to u. The calculations are straightforward, and will be omitted. The result is another equation of the same form as (6) for x but with the independent variable u. Three of the coefficients $q_1, \ldots, q_{n+1}$ of the new equation are found to be given by the formulas

$$(11) \qquad \begin{cases} u'q_1=p_1+n\eta/2 & (\eta=u''/u') , \\ u'^2q_2=p_2+(n-1)\eta p_1+(3n^2-5n+2)\eta^2/12+(n+1)\eta'/3 , & \\ \cdots \quad \cdots & \\ u'^{n+1}q_{n+1}=p_{n+1} . & \end{cases}$$

Since the transformation (10) is merely a change of parameter from t to u, the curve C is not changed. Hence we arrive at the result:

Equation (6) *and any equation into which it can be transformed by a transformation of parameter* (10) *have the same integral curves.*

It is now easy to reduce equation (6) to an especially simplified or *canonical* form. The effect of the total transformation (7), (10) on the functions p_1 and P_2 is found, by using first formulas (8) and then (11), to be given by the equations

$$(12) \qquad \begin{cases} u'\bar{p}_1=p_1+\lambda'/\lambda+n\eta/2 , \\ u'^2\bar{P}_2=P_2+(n+2)(\eta^2/2-\eta')/6 , \end{cases}$$

in which dashes indicate the transformed functions. Therefore *the functions λ and η can be chosen so that $\bar{p}_1=\bar{P}_2=0$.* For this purpose it is sufficient to choose η as a solution of the equation of Riccati,

$$\eta'=\eta^2/2+6P_2/(n+2) ,$$

and then to choose λ so that

$$\lambda'/\lambda = -p_1 - n\eta/2\ .$$

The form of equation (6) thus obtained is called *the Laguerre-Forsyth canonical form.* This form is characterized by the conditions $p_1 = p_2 = 0$ which, by the first of equations (9), are equivalent to $p_1 = P_2 = 0$.

We shall now define certain *projective invariants and covariants* of a curve and indicate briefly how they are connected with the geometry of the curve. A function of the coefficients of equation (6), and of their derivatives, which is invariant under the total transformation (7), (10), in the sense that it is unchanged by this transformation except possibly for a factor depending only on the transformation, is called a *projective invariant* of an integral curve C. *Every absolutely invariant equation connecting these invariants is independent of the analytic representation of the curve C and hence expresses a projective geometric property of C; conversely, every such property can be so expressed.* A function not only of the coefficients and their derivatives but also of x and the derivatives of x, which is invariant under the transformation (7), (10) in the sense just indicated, is spoken of as a *projective covariant* of the integral curve C. Every such covariant takes $n+1$ values when the $n+1$ coordinates x are substituted therein; and when these values are interpreted as the coordinates of a point *the covariant defines a curve whose points are in one-to-one correspondence with the points of the curve C, and which is obtainable from C by a projective geometric construction. Conversely, every such curve can be so represented analytically.* Therefore *the projective differential geometry of a curve can be studied by means of its invariants and covariants.* The method of studying a curve thus suggested is that which was used by Wilczynski (see Exs. 1, 3).

3. Linear osculants. Let us consider a curve C immersed in a linear space of n dimensions S_n. A fundamental problem is to find, for each positive integer k which is less than n, the linear subspace S_k of S_n which, among all linear subspaces with dimensions k, approximates the curve C most closely at a given point x on C. This subspace, which will be defined more precisely later on in this section, is *the osculating space* S_k of the curve C at the point x. Since a linear space S_k is determined by $k+1$ independent points, it will be found that the osculating space S_k at an ordinary point x of the curve C intersects C in $k+1$ "consecutive" points at the point x, and that the space S_k is consequently determined by x and the first k derivatives of x. Among these osculants the *tangent line*, the *osculating plane*, and the *osculating hyperplane* receive special consideration.

We begin with the classical definition of the tangent line. *The tangent line S_1 at an ordinary point P_x of a curve C is the limit of the straight line through P_x and a neighboring point P_1 on C as P_1 approaches P_x along C.* If the point P_x in Figure 1 corresponds to a value t of the parameter of the analytic

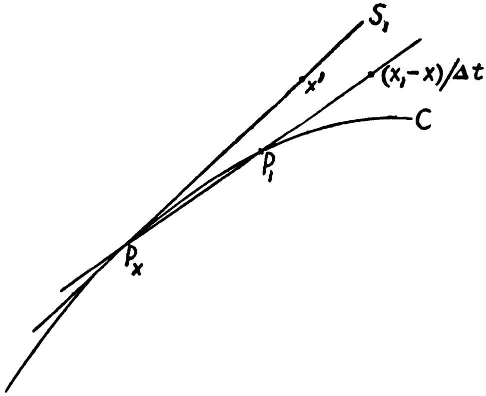

FIG. 1

curve C, and if P_1 corresponds to a value $t+\Delta t$, then the coordinates x_1 of P_1 can be represented by Taylor's expansion as power series of the form

$$x_1 = x + x'\Delta t + x''\Delta t^2/2 + \cdots . \tag{13}$$

The point defined by the expression

$$(x_1 - x)/\Delta t$$

is on the secant line P_1P_x, since this expression is a linear combination of x_1 and x. The limit of this point, namely the point x', is on the tangent line of the curve C at the point P_x, and is ordinarily distinct from P_x, coinciding with P_x only if P_x is fixed or else is a *stationary point* (cusp) of the curve C. So we reach the following result:

The tangent line at an ordinary point x of a proper analytic curve C is determined by the points x and x'.

Since $(\lambda x)' = \lambda' x + \lambda x'$, it follows that *the coordinates x can be multiplied by such a common factor λ that, after the multiplication, the derivative point will coincide with any point whatever on the tangent, except the point x itself.* In fact, if it is desired that the derivative point shall coincide with a point $x' + hx$, it is sufficient to choose λ so that $\lambda' = h\lambda$.

Geometers habitually employ elliptical language to express the idea of the limiting process employed in the definition of the tangent line. Thus the tangent line at an ordinary point P of a curve C is sometimes said to intersect C in two "consecutive" points at P. And sometimes any line that cuts a curve C in two "coincident" points at a point P, whether they are consecutive or not, is said to be tangent to C at P. For example, if a plane curve has an ordinary node, any line in the plane of the curve and passing through the node intersects the curve in two coincident points at the node and is called a tangent of the curve. Then *the two nodal tangents*, or *double-point* tangents, are such that each of them cuts the curve in three coincident points at the node, two of them being consecutive points on one branch of the curve, and one lying on the other branch.

Let us next consider the osculating plane, defined as follows. *The osculating plane S_2 at an ordinary point P_x of a curve C is the limit of the plane determined by P_x and two neighboring points P_1, P_2 on C as P_1, P_2 independently approach P_x along C.* Just as the tangent line is said to intersect the curve C in two consecutive points at P_x, so the osculating plane is said to intersect the curve in three consecutive points at P_x. Therefore *the osculating plane contains the tangent line.* The point defined by the expression

$$2(x_2-x-x'\Delta t)/\Delta t^2$$

lies in the plane containing the tangent line of C at P_x and also containing a neighboring point P_2 on C, since this expression is a linear combination of x, x', and x_2. The limit of this point, as P_2 approaches P_x, is shown by a series of the same form as (13) to be the point x'', which therefore lies in the osculating plane of the curve C at the point P_x. The point x'' ordinarily does not lie on the tangent line, being on it only if the curve C is a straight line or else has an *inflexion* at the point P_x. So we reach the following theorem:

The osculating plane at an ordinary point x of a proper curve C not a straight line is determined by the points x, x', x''.

The osculating space S_k at a point of a curve will now be defined precisely. *The osculating linear space S_k at an ordinary point P of a curve C immersed in space $S_n (0<k<n)$ is the limit of the space S_k determined by P and k neighboring points on C, as each of these points independently approaches P along C.* Briefly, this is the space S_k that intersects C in $k+1$ consecutive points at P. Continuation of the reasoning used above leads to the following theorem:

At an ordinary point x of an analytic curve C immersed in a linear space S_n the osculating linear space $S_k (0<k<n)$ of C is determined by the points

$$x, x', x'', \ldots, x^{(k)}.$$

The configuration composed of a point x of a curve C and the osculating linear spaces $S_1, S_2, \ldots, S_r$ of C at this point is called *the element E_r of the curve C at the point x.* The element E_r not only has as one of its component parts the osculating space S_r, but geometrically is situated in this osculating space.

We conclude this section by proving a theorem concerning a one-parameter family of hyperplanes. The osculating hyperplanes of a curve obviously form a one-parameter family. Conversely, it can be shown that *the hyperplanes of an analytic one-parameter family osculate a curve.* For this purpose let us consider a hyperplane,

$$\sum_{i=1}^{n+1} \xi_i x_i = 0 \; , \tag{14}$$

whose $n+1$ coordinates ξ are analytic functions of a variable t. Differentiating $n-1$ times we obtain, dropping the limits from the summation signs,

$$\Sigma \xi' x = 0, \ldots, \Sigma \xi^{(n-1)} x = 0 \; .$$

Solving these $n-1$ equations and (14) for the ratios of the coordinates x, and indicating a determinant by writing a typical row within parentheses, we obtain the parametric vector equation

$$x = (\xi, \xi', \ldots, \xi^{(n-1)}) \tag{15}$$

of a curve. The osculating hyperplane of this curve can be shown to be the hyperplane (14) by verifying that not only the x in equation (15) but also the $x', \ldots, x^{(n-1)}$ computed* from (15) satisfy equation (14). Thus the proof is complete. Incidentally, we note that in particular the planes of a one-parameter family in ordinary space osculate a curve, and that the straight lines of a one-parameter family in a plane envelop a curve.

4. Plane curves. The projective differential geometry of plane curves was first systematically studied by Halphen in his Paris doctoral thesis of 1878, and most of the results of this section are due to him. Together with the tangent line and the osculating conic the cubics having eight-point contact at a point of a plane curve are especially significant. Among these cubics *the eight-point nodal cubic* and *the osculating,* or *nine-point, cubic* attract attention. Consideration of these osculants leads to a canonical ex-

* The derivative of a determinant of order n can be expressed as the sum of n such determinants by differentiating column at a time and adding results, thus: $(x, y, z)' = (x', y, z) + (x, y', z) + (x, y, z')$. See Pascal, *I determinanti* (2d ed.; Milano: Ulrico Hoepli, 1923), p. 72.

pansion of one non-homogeneous coordinate of a point on a plane curve as a power series in the other coordinate. The coordinate system for this expansion has a purely geometric description, with which this section closes.

An algebraic plane curve may be defined as the locus of a point whose coordinates x_1, x_2, x_3 satisfy the equation that results from setting a polynomial in these coordinates equal to zero. *The order* of an algebraic plane curve is defined to be the number of points in which any straight line not a component of the curve intersects it. This number is equal to the degree of the equation of the curve. For example, a straight line, when regarded as a plane curve, is of order one. Of all algebraic curves *immersed* in a plane, the conic is the curve of lowest order, namely, two.

In analytic projective geometry it is shown that *the equation of any non-composite conic* can be written in the form

$$x_1x_3 - x_2^2 = 0$$

by choosing the coordinate system so that the conic passes through the vertex (1, 0, 0) of the triangle of reference tangent to the side $x_3 = 0$, through (0, 0, 1) tangent to $x_1 = 0$, and through the unit point. If non-homogeneous coordinates are introduced by the definitions

$$x = x_2/x_1\,, \qquad y = x_3/x_1\,, \tag{16}$$

the equation of the conic becomes

$$y = x^2\,. \tag{17}$$

We shall sometimes use homogeneous coordinates, and sometimes non-homogeneous, according to convenience, the notation in each case indicating clearly which coordinates are being used.

The equation of any analytic plane curve C can be written in the form of a power series expansion,

$$y = a_0 + a_1x + a_2x^2 + \cdots. \tag{18}$$

This series represents the curve C in the neighborhood of the point P whose coordinates are 0, a_0, it being understood that x, y are the coordinates of a variable point in the neighborhood of P on C, and that the neighborhood is so small that the series converges. We shall obtain a canonical form for the expression (18) by choosing the coordinate system so that it will be covariantly, or geometrically, connected with the curve C.

The osculating conic at a point of a plane curve is defined as follows. *The*

osculating conic at a point P of a plane curve C is the limit of the conic determined by P and four neighboring points on C, as these points independently approach P along C. In order to find the equation of the osculating conic K at the point $P(0,\ a_0)$ of the curve C represented by equation (18), it is sufficient to write the most general quadratic equation in x, y and to demand that this equation be satisfied by the power series for y in equation (18) identically in x as far as the terms in x^4. Let us suppose that the coordinate system has been chosen so that the equation of the osculating conic K has the form (17). Then it turns out that we must have $a_0=a_1=a_3=a_4=0$, $a_2=1$. The equation of the curve C can thus be written in the form

$$y=x^2+ax^5+a_6x^6+a_7x^7+\cdots. \tag{19}$$

It will be observed that this equation actually coincides with the equation (17) of the osculating conic as far as the terms in x^4, and that the coordinates of the point P are now 0, 0. Clearly, *the curve C and its osculating conic K have at the point P the same tangent line, $y=0$.*

Ordinarily we have $a\neq 0$. For, *if $a=0$, then the conic K hyperosculates the curve C at the point P;* that is, K has more than five consecutive points in common with C at P, and P is a singular point. Such a singular point is called a *sextactic* point. If every point of the curve C is a sextactic point, the osculating conic is the same at every point of C, and the curve C is therefore itself a conic. We shall as a rule exclude sextactic points from further consideration hereinafter.

There is a pencil of cubic curves each of which has eight-point contact with the curve C at the point P. The equation of a general one of these ∞^1 eight-point cubics can be obtained by writing the most general cubic equation in x, y and demanding that this equation be satisfied by the power series for y in equation (19) identically in x as far as the terms in x^7. The result is

$$a(x^3+ay^3-xy)+a_6(y-x^2)y+h[a(y-x^2-axy^2-a_6y^3)-a_7(y-x^2)y]=0\ ,$$

where h is an arbitrary constant. The only one of these cubics that has a node at the point P is found, by demanding that the derivative dy/dx be indeterminate at (0, 0), to be the one for which $h=0$. Its nodal tangents are found, by equating to zero the terms of lowest degree in its equation, to be the lines

$$y=0, \qquad ax-a_6y=0\ .$$

If the triangle of reference is chosen so that the latter line is the side $x=0$, then $a_6=0$ and *the equation of the eight-point nodal cubic is*

$$x^3+ay^3-xy=0\ , \tag{20}$$

while the equation of the curve C takes the form

$$y=x^2+ax^5+bx^7+a_8x^8+\cdots. \tag{21}$$

It is known that every nodal cubic has three collinear inflexions. These can be found* by making the equation of the cubic homogeneous and solving it simultaneously with its hessian, the solution which gives the double point being discarded. Thus we find that *the inflexions of the cubic* (20) *lie on the line* $x_1=0$, *some one of them lying on each of the lines*

$$x_2^3+ax_3^3=0 .$$

The branch of the eight-point nodal cubic which is tangent to the curve C at the point P is found to be represented by the expansion

$$y=x^2+ax^5+3a^2x^8+\cdots,$$

and therefore intersects C in seven consecutive points at P. The other branch is represented by

$$x=ay^2+a^3y^5+3a^5y^8+\cdots,$$

and therefore merely intersects C in one point at P.

It is furthermore known that all plane cubic curves through eight given points have also a ninth point in common†; it follows that all of the eight-point cubics,

$$x^3+ay^3-xy+h[a(y-x^2-axy^2)-by(y-x^2)]=0 , \tag{22}$$

at the point P of the curve C have a ninth point in common. This point, which is usually distinct from P, is called‡ *the Halphen point* corresponding to the point P of the curve C, and its homogeneous coordinates are found, by solving equations (20) and (22) simultaneously for x and y, to be

$$(a^5+b^3,\ a^3b,\ ab^2) .$$

Since we are supposing $a\neq 0$, this point coincides with the point P in case $b=0$, and then P is called *a coincidence point.* A curve all of whose points are coincidence points is called *a coincidence curve.*

* Salmon, 1879. 1, p. 59.

† *Ibid.*, p. 18.

‡ Wilczynski, 1906. 1, p. 68. Halphen, 1918. 4, p. 200; Paris thesis of 1878, entitled *Sur les invariants différentiels.*

Among the eight-point cubics there is a nine-point, or osculating, cubic whose equation can be obtained by demanding that the power series for y in equation (21) satisfy equation (22) identically in x as far as the terms in x^8. Thus we obtain the condition on h,

$$b+ah(2a^2-a_8)=0 .$$

Let us complete the characterization of the coordinate system by choosing the unit point as the point, distinct from P, where the osculating conic intersects the osculating cubic. Since the unit point is already on the osculating conic we now merely demand that it be also on the osculating cubic, and find that $ah=1$ and hence $a_8=b+2a^2$. Thus the equation of the curve C becomes

$$y=x^2+ax^5+bx^7+(b+2a^2)x^8+\cdots \qquad (a\neq 0) , \tag{23}$$

and the equation of the osculating cubic is

$$a(x^3+ay^3-xy+y-x^2-axy^2)-by(y-x^2)=0 .$$

So we reach the following conclusion, illustrated by Figure 2.

The projective differential geometry of a plane curve C in the neighborhood of an ordinary point P on C, can be studied by means of the expansion (23), *all of whose coefficients are absolute invariants of the curve. The covariant coordinate system for this expansion has the following geometric description. One vertex of the triangle of reference is the point $P(1, 0, 0)$ under consideration. One side, $x_3=0$, is the tangent line of C at P. The other side, $x_2=0$, through P is the nodal tangent of the eight-point nodal cubic distinct from the tangent of C; and the third side, $x_1=0$, is the line of inflexions of the eight-point nodal cubic. The unit point is the point distinct from P common to the osculating conic and the osculating cubic at the point P of the curve C.*

5. Cubic curves in ordinary space. In ordinary space S_3 *an algebraic surface* may be defined as the locus of a point whose coordinates $x_1, \ldots, x_4$ satisfy the equation that results when a polynomial in these coordinates is set equal to zero. *The order* of such an algebraic surface is defined to be the number of points in which any straight line that does not lie entirely on the surface intersects it. This number is equal to the degree of the equation of the surface. A plane, when regarded as a surface in space S_3, is of order one. Of all algebraic surfaces *immersed* in ordinary space the quadric is the surface of lowest order, namely, two.

An algebraic curve in ordinary space is by definition all, or part, of the intersection of two algebraic surfaces, and *the order* of such a curve is de-

fined to be the number of points in which it intersects any plane not containing a component of it. Of all algebraic curves *immersed* in ordinary space, the curve of lowest order is *the twisted cubic.* This curve of order three will be studied in this section to prepare the way for the theory of unspecialized analytic curves in space S_3, which is to follow.

The simplest possible parametric representation of a twisted cubic will first of all be deduced from the definition of the curve, and then the determination of the cubic by six of its points will be discussed. Considerations

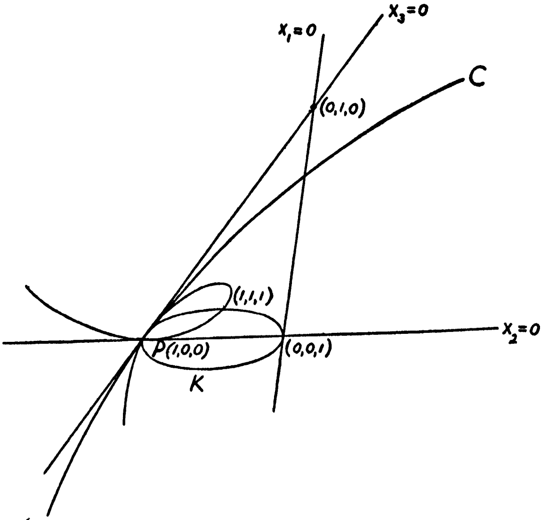

FIG. 2

of the tangent lines and the osculating planes of the cubic will lead to the introduction of *the osculating conic* at a point of the curve, and to the definition of the correspondence between points and planes which is known as *the null system* of the twisted cubic.

In ordinary space *a twisted cubic curve is defined to be the residual intersection of two non-composite quadric cones that have one, and only one, generator in common* (see Ex. 16). The parametric equations of such a curve may be found in the following way. If a quadric cone has its vertex at the point $(1, 0, 0, 0)$; if this cone is tangent to the plane $x_4=0$ along the line $x_4=x_3=0$, and to $x_2=0$ along $x_2=x_3=0$; and if this cone contains the unit point, then its equation is

$$x_3^2-x_2x_4=0\,. \tag{24}$$

Similarly, if a cone has its vertex at the point (0, 0, 0, 1); if this cone is tangent to the plane $x_1=0$ along the line $x_1=x_2=0$, and to $x_3=0$ along $x_3=x_2=0$; and if this cone contains the unit point, then its equation is

$$x_2^2-x_1x_3=0 \,. \tag{25}$$

These two cones obviously have in common the generator $x_2=x_3=0$. The remainder of their intersection is the twisted cubic whose equations in terms of a parameter t are

$$x_1=1 \,, \qquad x_2=t \,, \qquad x_3=t^2 \,, \qquad x_4=t^3 \,. \tag{26}$$

Thus the following theorem is proved:

By suitable choice of coordinate system and parameter, any twisted cubic curve in ordinary space can have its parametric equations written in the form (26).

It is not difficult to demonstrate the truth of the following statement. *A twisted cubic is determined by six points $P_1, \ldots, P_6$ no four of which are coplanar.* For, there is a unique quadric cone with vertex at P_1 and containing $P_2, \ldots, P_6$; and there is another unique quadric cone with vertex at P_6 and containing $P_1, \ldots, P_5$. These cones have the line P_1P_6 in common, and the remainder of their intersection is a unique twisted cubic* through the six given points.

In order to introduce *the osculating conic* at a point of a twisted cubic we begin with the following remarks. *The twisted cubic* (26) *passes through the point* (1, 0, 0, 0) *tangent to the line* $x_3=x_4=0$, since the curve has two coincident points in this line. Moreover, *the osculating plane of the cubic at this point is the plane* $x_4=0$, since the curve has three coincident points in this plane. The tangent line at the point x of the cubic that corresponds to any value of t is known to be determined by the points x, x', the accent indicating differentiation with respect to t; therefore this line has the parametric equations

$$x_1=1 \,, \qquad x_2=t+\lambda \,, \qquad x_3=t(t+2\lambda) \,, \qquad x_4=t^2(t+3\lambda) \,,$$

where λ is a parameter. This line meets the plane $x_4=0$ in the point whose coordinates are given by

$$x_1=1 \,, \qquad x_2=2t/3 \,, \qquad x_3=t^2/3 \,, \qquad x_4=0 \,.$$

* Snyder and Sisam, 1914. 1, p. 230.

The locus of this point, as t varies, is a conic called *the osculating conic of the cubic at the point* (1, 0, 0, 0). The algebraic equations of this conic are

$$4x_1x_3-3x_2^2=x_4=0 . \tag{27}$$

A remarkable property of the osculating planes of a twisted cubic will now be explained, and *the null system* of the cubic will be defined. The osculating plane at any point x of the twisted cubic (26) is known to be determined by the points x, x', x''; therefore this plane has the equation

$$t^3x_1-3t^2x_2+3tx_3-x_4=0 ,$$

as can be readily verified by means of equations (26) and the equations obtained therefrom by differentiating twice with respect to t. Through any point y of the space S_3 there pass three of these planes, whose points of osculation have parameter values t_1, t_2, t_3, connected by the relations

$$t_1+t_2+t_3=3y_2/y_1 , \qquad t_1t_2+t_2t_3+t_3t_1=3y_3/y_1 , \qquad t_1t_2t_3=y_4/y_1 .$$

These three points of osculation determine the plane

$$y_4x_1-3y_3x_2+3y_2x_3-y_1x_4=0 ,$$

which evidently also passes through the point y. The coordinates ξ of this plane are given by the equations

$$\xi_1=y_4 , \qquad \xi_2=-3y_3 , \qquad \xi_3=3y_2 , \qquad \xi_4=-y_1 . \tag{28}$$

So *a twisted cubic determines a one-to-one correspondence between the points and planes of ordinary space. When a point is given, the corresponding plane is the plane containing the three points of osculation of the three osculating planes of the cubic that can be drawn through the given point. Corresponding point and plane are in united position. The equations* (28) *of the correspondence, being linear, show that it is in fact a correlation.* Since a correlation in ordinary space which is such that corresponding point and plane are in united position is customarily called a *null system,* we make the following definition. *The correlation* (28) *is the null system of the twisted cubic* (26).

If non-homogeneous coordinates are introduced by the definitions

$$x=x_2/x_1 , \qquad y=x_3/x_1 , \qquad z=x_4/x_1 , \tag{29}$$

the equations of the twisted cubic become

$$y=x^2 , \qquad z=x^3 , \tag{30}$$

and then the equations of the osculating conic are

$$4y-3x^2=z=0\,. \tag{31}$$

6. Curves in ordinary space. The projective differential geometry of curves in ordinary space was first studied systematically by Halphen in a memoir published in 1880. With the notable exceptions of the point first defined by Sannia in 1926 and now called *the point of Sannia,* and of *the osculating conic,* which was introduced by Wilczynski in 1905, the contents of this section are, for the most part, found in the memoir* of Halphen.

In ordinary space an analytic curve can be defined by expressing two of the non-homogeneous coordinates of a point on the curve as power series in the third coordinate. Some of the more interesting osculants at a point of such a curve are the tangent line and the osculating plane, *the osculating twisted cubic, the osculating conic,* and *the osculating quadric cone. The point of Sannia, the point of Halphen,* and *the principal plane* are associated with the point of the curve. Consideration of the various osculants and associated configurations yields at the close of this section a geometric description of a coordinate system for which the equations of the curve assume an especially simple canonical form.

In non-homogeneous coordinates the equations of any analytic curve C in ordinary space S_3 can be written in the form of two power series expansions,

$$\begin{cases} y=a_0+a_1x+a_2x^2+\cdots\,, \\ z=c_0+c_1x+c_2x^2+\cdots\,. \end{cases} \tag{32}$$

These series represent the curve C in a sufficiently small neighborhood of the point P whose coordinates are $0,\ a_0,\ c_0$. Since a twisted cubic is determined by six of its points we adopt the following definition. *The osculating twisted cubic at the point P of the curve C is the twisted cubic having six-point contact with C at P.* Let us suppose that the coordinate system has been chosen so that the osculating cubic at the point $P(0,\ a_0,\ c_0)$ of the curve C has the equations (30). Then equations (32) must coincide with equations (30) as far as the terms in x^5; thus the equations of the curve C become

$$y=x^2+a_6x^6+\cdots\,, \qquad z=x^3+c_6x^6+\cdots\,, \tag{33}$$

the coordinates of the point P being now $0,\ 0,\ 0$. *The curve C and its osculating cubic have at the point P the same tangent line, $y=z=0$, and the same osculating plane, $z=0$.* The osculating conic (31) of the osculating cubic of

* Halphen, 1880. 1.

the curve C at the point P is sometimes called* *the osculating conic of C at P,* although it has only two-point contact with the curve C and the cubic.

There is a bundle of quadric surfaces each of which has seven-point contact with the curve C at the point P. The equation of a general one of these ∞^2 seven-point quadrics can be obtained by writing the most general equation of the second degree in x, y, z and imposing on it the condition that it be satisfied by the power series (33) for y and z identically in x as far as the terms in x^6. The result is

$$h(y-x^2-a_6z^2)+k(y^2-zx)+l(z-xy-c_6z^2)=0 ,$$

where h, k, l are arbitrary constants.

Since all quadrics through seven given points are known to have also an eighth point in common,† it follows that all of the seven-point quadrics at the point P of the curve C have an eighth point in common. This point is called‡ *the point of Sannia* corresponding to the point P of the curve C, and its homogeneous coordinates are found to be

$$(a_6^3+c_6^4,\ a_6^2c_6,\ a_6c_6^2,\ c_6^3) .$$

If $c_6\neq 0$ let us further restrict the coordinate system by imposing on it a condition which may at first sight appear rather artificial; precisely, let us *choose the point of Sannia for the point* $(1, 0, 0, 1)$. Then $a_6=0$, $c_6=1$. Therefore *the equations of the curve C* can be written in the form

$$(34) \qquad y=x^2+ax^7+bx^8+\cdots, \qquad z=x^3+x^6+cx^7+dx^8+\cdots,$$

while *the equation of the seven-point quadrics* reduces to

$$(35) \qquad h(y-x^2)+k(y^2-zx)+l(z-xy-z^2)=0 .$$

It will be shown presently that the coordinate system is essentially determined by the conditions that we have imposed, so that all of the coefficients of the series (34) are absolute invariants of the curve C. The geometrical significance of the hypothesis made above, that $c_6\neq 0$, will be explained at the end of the next section.

Among the seven-point quadrics there are ∞^1 cones, for which the discriminant of (35) vanishes, so that h, k, l satisfy the condition

$$(hk+l^2)^2-4h^3l=0 .$$

* Wilczynski, 1905. 1, p. 112.

† Snyder and Sisam, 1914. 1, p. 167.

‡ Fubini and Čech, 1926. 1, p. 43; Sannia, 1926. 6, p. 18.

The only one of these cones that has its vertex at the point P is the one for which $h=l=0$, as may be verified by making equation (35) homogeneous and then demanding that the four first partial derivatives of the left member vanish at (1, 0, 0, 0). This cone, whose equation is

$$y^2-zx=0\ , \tag{36}$$

is called *the osculating quadric cone* at the point P of the curve C. *The locus of the vertices of all the seven-point cones is a rational curve of the sixth order* whose parametric equations can be written in the form

$$x_1=1-3t^3\ ,\qquad x_2=t^5-t^2\ ,\qquad x_3=t^4\ ,\qquad x_4=-t^6\ , \tag{37}$$

where $t^2=h/l$.

There is a second bundle of quadric surfaces associated with a point P of a curve C. If a quadric surface passes through the osculating cubic (30) at the point P of the curve C, the equation of the quadric has the form

$$h(y-x^2)+k(y^2-zx)+l(z-xy)=0\ , \tag{38}$$

since the equation of such a quadric must be satisfied by the expressions for y and z in equations (30) identically in x. The totality of such quadrics is evidently a bundle. Among these ∞^2 quadrics there are ∞^1 cones, for which $hk+l^2=0$. The vertices of these cones lie on the osculating cubic. Among these ∞^1 cones there are only two that have seven-point contact with the curve C at the point P; for them $l=hk=0$. The one of these for which we have $l=h=0$ is the osculating quadric cone (36) with its vertex at the point P. The other, for which $l=k=0$, has the equation

$$y-x^2=0\ ,$$

and its vertex is at the point (0, 0, 0, 1). *This point is called* the Halphen point corresponding to the point P of the curve C. It is the only point distinct from P from which the osculating cubic at the point P of the curve C can be projected by a quadric cone having seven-point contact with C at P.*

The principal plane at a point of a curve next engages our attention. If the curve C and its osculating cubic at a point P are projected from a point onto their common osculating plane at P, the projections ordinarily have contact of the same order as that of C and the cubic, namely, fifth-order, or six-point, contact. We shall prove in the next paragraph that *the projections have seven-point contact if, and only if, the center of projection is in the plane $y=0$. This plane is called the principal plane at the point P of the curve*

* Fubini and Čech, 1926. 1, p. 42.

C. This result is a special case of the more general theorem* of Halphen, that if two curves have at a point contact of order n, there is one plane, called the principal plane, containing the common tangent line of the two curves at this point, such that the projections of the two curves from any point in this plane onto any other plane have contact of order $n+1$.

Let us proceed to the proof promised in the last paragraph. The equations of the straight line joining any point (x, y, z) of the curve C and any point (α, β, γ), which we shall suppose not to be in the osculating plane $\zeta=0$, are

$$(39) \qquad \xi=\alpha+(x-\alpha)t\,, \qquad \eta=\beta+(y-\beta)t\,, \qquad \zeta=\gamma+(z-\gamma)t\,,$$

wherein y, z are given by equations (34) and t is a parameter. The equations of the projection of C from (α, β, γ) onto the plane $\zeta=0$ are found by using $t=-\gamma/(z-\gamma)$ in equations (39). The result is

$$(40) \qquad \begin{cases} \xi=x-\alpha x^3/\gamma+x^4/\gamma-\alpha(1+\gamma)x^6/\gamma^2+\cdots\,, \\ \eta=x^2-\beta x^3/\gamma+x^5/\gamma-\beta(1+\gamma)x^6/\gamma^2+\cdots\,. \end{cases}$$

Inverting† the first of these series to obtain x as a power series in ξ, we find

$$x=\xi+\alpha\xi^3/\gamma-\xi^4/\gamma+3\alpha^2\xi^5/\gamma^2+\cdots\,;$$

and substituting the result in the second of (40), we obtain *the equation of the projection of the curve C onto its osculating plane*, $\zeta=0$, namely

$$(41) \quad \eta=\xi^2-\beta\xi^3/\gamma+2\alpha\xi^4/\gamma-(\gamma+3\alpha\beta)\xi^5/\gamma^2+(7\alpha^2+2\beta-\beta\gamma)\xi^6/\gamma^2+\cdots\,.$$

If we suppose that y, z, in equations (39) are given by (30) instead of by (34), and repeat the operations just outlined, we arrive at the equation of the projection of the osculating cubic onto the osculating plane; this equation is found to be the same, to terms of order six, as equation (41) except that in the coefficient of ξ^6 the term $\beta\gamma$ is missing. Therefore the projections have seven-point contact if, and only if, $\beta=0$, since $\gamma\neq 0$; in this case the center of projection lies in the plane $y=0$, as was to be proved.

The principal point of the tangent line at a point of a curve has the following definition. *The point corresponding to the principal plane, $y=0$, in the null system of the osculating cubic is*‡ *the principal point of the tangent.* Its

* Halphen, 1880. 1, p. 25; 1918. 4, pp. 375–76.

† Goursat-Hedrick, 1904. 1, p. 406.

‡ Wilczynski, 1906. 1, p. 271.

homogeneous coordinates are found by use of equations (28) to be (0, 1, 0, 0,); so *the principal point lies on the tangent line,* $x_3=x_4=0$.

The coordinate system for the expansions (34) can now be visualized as in Figure 3 and described geometrically as follows. The vertex (1, 0, 0, 0) of the tetrahedron of reference is the point P of the curve under considera-

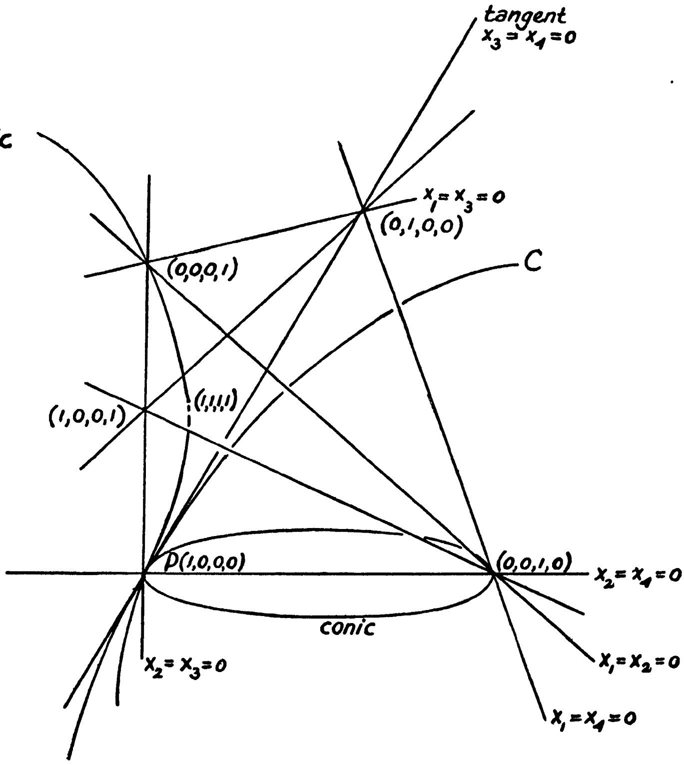

FIG. 3

tion, and the edge $x_3=x_4=0$ is the tangent of the curve C at the point P. The face $x_4=0$ is the osculating plane, and the face $x_3=0$ is the principal plane, of C at P. The vertex (0, 1, 0, 0) is the principal point of the tangent. The edge $x_2=x_4=0$ is the polar line of the principal point with respect to the osculating conic (27) of C at P; and the vertex (0, 0, 1, 0) is the point distinct from P where this line meets the osculating conic. The vertex (0, 0, 0, 1) is the Halphen point of C at P. Thus the tetrahedron is char-

acterized. The unit point is one of the three points where the osculating cubic intersects the plane $x_1 - x_4 = 0$ determined by the vertex (0, 0, 1, 0), the point of Sannia (1, 0, 0, 1), and the principal point (0, 1, 0, 0).

7. The osculating linear complex of a curve in S_3. We shall frequently have occasion hereinafter to consider certain configurations of lines. It is known that all the straight lines of ordinary space S_3 form a four-parameter family. A three-parameter family of these lines is called a *line complex*, or simply a *complex;* a two-parameter family is called a *congruence;* and a one-parameter family, a *ruled surface.*

After the plückerian homogeneous coordinates of a line in ordinary space are introduced early in this section it is easy to define a *line complex* analytically and in particular a *linear line complex.* The *null system* of a linear complex appears next for brief consideration. *The osculating linear complex* at a point of a curve is then defined and its equation, referred to the coordinate system of the preceding section, is calculated. Finally, the conditions are obtained that the osculating linear complex at a point of a curve may hyperosculate the curve at the point, and that a curve may belong to a linear complex.

We proceed to introduce the plückerian coordinates of a line. If $y_1, \ldots, y_4$ and $z_1, \ldots, z_4$ are homogeneous coordinates of two distinct points P_y and P_z on a line l in ordinary space, then *the plückerian homogeneous coordinates* of l are defined to be the six numbers $\omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{42}, \omega_{34}$ given by

$$\omega_{ik} = y_i z_k - y_k z_i \qquad (i, k = 1, \ldots, 4) \,. \tag{42}$$

It is clear that only the ratios of these coordinates are determined when the points y and z are given; and it may be verified that, if any other pair of points on the line l is used in place of y, z, essentially the same line coordinates result. Moreover, these coordinates satisfy the equation

$$\omega_{12}\omega_{34} + \omega_{13}\omega_{42} + \omega_{14}\omega_{23} = 0 \,, \tag{43}$$

as can be verified by direct substitution. Conversely, in treatises on line geometry it is shown that any six numbers satisfying this relation can be regarded as the coordinates of a line.

We next state some fundamental definitions. *The locus of a line whose coordinates satisfy a homogeneous equation with constant coefficients is a complex. If the equation is linear, of the form*

$$a_{34}\omega_{12} + a_{42}\omega_{13} + a_{23}\omega_{14} + a_{14}\omega_{23} + a_{13}\omega_{42} + a_{12}\omega_{34} = 0 \,, \tag{44}$$

the complex is called a linear complex. Substituting in this equation the expressions for the coordinates ω_{ik} given in (42), and collecting coefficients of z, we obtain an equation of the form $\Sigma\xi z=0$ whose coefficients ξ are defined, except for a non-vanishing proportionality factor ρ, by the equations

$$
\begin{cases}
\rho\xi_1 = \phantom{a_{34}y_1} - a_{34}y_2 - a_{42}y_3 - a_{23}y_4 \,, \\
\rho\xi_2 = a_{34}y_1 \phantom{- a_{34}y_2} - a_{14}y_3 + a_{13}y_4 \,, \\
\rho\xi_3 = a_{42}y_1 + a_{14}y_2 \phantom{- a_{14}y_3} - a_{12}y_4 \,, \\
\rho\xi_4 = a_{23}y_1 - a_{13}y_2 + a_{12}y_3 \,.
\end{cases} \tag{45}
$$

If the point P_y is held fixed and P_z is allowed to vary, we see that *all of the lines of a linear complex that pass through a fixed point lie in a fixed plane which passes through the point, and therefore form a flat pencil with center at the point.* So a linear complex determines a correspondence between the points and planes of ordinary space, which is, in fact, a correlation with corresponding point and plane in united position. This correlation determined by a linear complex and represented analytically by equations (45) is called *the null system of the complex.*

Since equation (44) contains five essential constants it follows that a linear complex is determined by five of its lines. Hence we make the following definition. *The osculating linear complex at a point P of a curve C is the limit of the complex determined by the tangent of C at P and the tangents at four neighboring points of C, as each of these points independently approaches P along C.* In order to find the equation of the osculating linear complex of the curve (34) at the point (0, 0, 0), we observe that, when non-homogeneous coordinates are introduced by the definitions (29), the coordinates of the line joining the points (x, y, z) and (ξ, η, ζ) are given by

$$
\begin{cases}
\omega_{12} = \xi - x \,, & \omega_{13} = \eta - y \,, & \omega_{14} = \zeta - z \,, \\
\omega_{23} = x\eta - y\xi \,, & \omega_{42} = z\xi - x\zeta \,, & \omega_{34} = y\zeta - z\eta \,.
\end{cases} \tag{46}
$$

The parametric equations of the tangent line of the curve (34) at the point (x, y, z) are

$$\xi = x + \lambda \,, \qquad \eta = y + \lambda y' \,, \qquad \zeta = z + \lambda z' \,,$$

where λ is the parameter and accents denote differentiation with respect to x. The line coordinates of this tangent are therefore given by

$$
\begin{cases}
\omega_{12} = 1 \,, & \omega_{13} = y' \,, & \omega_{14} = z' \,, \\
\omega_{23} = xy' - y \,, & \omega_{42} = z - xz' \,, & \omega_{34} = yz' - zy' \,.
\end{cases} \tag{47}
$$

Using the series (34) and the expressions obtained therefrom for y', z', we get, to terms of the fourth order,

$$\begin{cases} \omega_{12}=1\,, & \omega_{13}=2x+\cdots\,, & \omega_{14}=3x^2+\cdots\,, \\ \omega_{23}=x^2+\cdots\,, & \omega_{42}=-2x^3+\cdots\,, & \omega_{34}=x^4+\cdots\,; \end{cases} \tag{48}$$

and demanding that equation (44) be satisfied by these expressions identically in x as far as the terms in x^4, we find *the equation of the osculating linear complex* at the point (0, 0, 0) of the curve (34), namely,

$$\omega_{14}-3\omega_{23}=0\,. \tag{49}$$

For this complex equations (45) reduce to (28). Therefore we have the theorem:

The null system of the osculating linear complex at a point P of a curve C is the same as the null system of the osculating twisted cubic of C at P.

We conclude by explaining the geometrical significance of the hypothesis $c_6\neq 0$, which was made in the last section. If in computing the equation of the osculating linear complex we had used equations (33) and the tetrahedron of reference described in connection therewith, instead of (34), the equation of this complex would still have turned out to be precisely (49). Computing the left member of (49) to terms of the fifth degree we find

$$\omega_{14}-3\omega_{23}=6c_6x^5+\cdots\,.$$

Therefore *a point at which $c_6=0$ is a point at which the osculating linear complex hyperosculates the curve.* Such singular points were excluded in deriving equations (34). If every point of a curve is such a point, the osculating linear complex is the same at every point of the curve, and the curve belongs to a linear complex, in the sense that its tangents are lines of the complex.

EXERCISES

1. The functions θ_3 and θ_8 of the coefficients of equation (5) and their derivatives, which are defined by placing

$$\theta_3=P_3-3P_2'/2\,,\qquad \theta_8=6\theta_3\theta_3''-7\theta_3'^2-27P_2\theta_3^2\,,$$

are relative invariants of weights 3 and 8 respectively under the transformation (10), and are absolute seminvariants. WILCZYNSKI, 1906. 1, p. 59

2. If equation (5) is in its Laguerre-Forsyth canonical form, and if a point $y_1x+y_2x'+y_3x''$ is said to have *local coordinates* y_1, y_2, y_3, show that the local coordinates of a point near the point P on the curve C are represented by

$$y_1=1-P_3\Delta t^3/6-P_3'\Delta t^4/24-\cdots, \qquad y_2=\Delta t-P_3\Delta t^4/24+\cdots,$$
$$y_3=\Delta t^2/2+\cdots.$$

Hence show that the local equation of the osculating conic of C at P is $y_2^2-2y_1y_3=0$.
WILCZYNSKI, 1906. 1, p. 62

3. Use the Laguerre-Forsyth canonical form of equation (5) to prove that if $\theta_3=0$, the integral curves of (5) are all conics. WILCZYNSKI, 1906. 1, p. 90

4. Use the result of Exercise 1 and the first of equations (12) for $n=2$ to obtain *the Halphen canonical form* of equation (5), for which $p_1=0$, $\theta_3=1$. What is the most general transformation (7), (10) preserving this canonical form? Prove that for this form the points x', x'' are covariant, and describe them geometrically.

5. Show that by a suitable transformation of unit point equation (23) becomes

$$y=x^2+x^5+Ax^7+Bx^8+\cdots,$$

where the coefficients A, B are given by

$$A=b/a^{5/3}, \qquad B=(b+2a^2)/a^2.$$

Describe the new unit point geometrically.

6. In the notation of equation (23) the cross ratio of the tangent of the curve at the point P, the other double-point tangent of the eight-point nodal cubic, the line from P to the Halphen point, and the line from P to the unit point is b/a^2.

7. Among the seven-point quadrics (35) there is a pencil of eight-point quadrics whose equation is

$$h[y-x^2+a(y^2-zx)]+l[z-xy-z^2+c(y^2-zx)]=0.$$

The one of these eight-point quadrics for which $h=0$ is the only one that contains the line $y=z=0$; and is the only one to which the line $x=z=0$ is tangent. The eight-point quadric for which $l=0$ is the only one containing the line $x=y=0$; is the only one which meets the line $x_1=z=0$ in points separating the points (0, 1, 0, 0) and (0, 0, 1, 0) harmonically; and is the only one containing the osculating cubic (30). For the osculating, or nine-point, quadric the ratio h/l satisfies the condition $h(b-ac)+l(d-a-c^2)=0$; and for the four eight-point cones this ratio satisfies the condition

$$a^2h^4+2(ac-2)h^3l+(2a+c^2)h^2l^2+2chl^3+l^4=0.$$

8. The twisted cubics cut on the osculating quadric cone (36) by the ∞^1 cones $y-x^2+hyz=0$ are the ∞^1 five-point cubics of the curve C at the point P; their parametric equations can be written in the form

$$x_1=1-ht^3\,,\qquad x_2=t\,,\qquad x_3=t^2\,,\qquad x_4=t^3\,.$$

All of these cubics have the same osculating conic at P, and have the same null system.

9. The coordinates of the osculating plane at the point P of a five-point cubic defined in Exercise 8 are given by

$$\xi_1=-t^3\,,\qquad \xi_2=3t^2\,,\qquad \xi_3=-3t\,,\qquad \xi_4=1-ht^3\,,$$

and satisfy the equations

$$\xi_2^2-3\xi_1\xi_3=0\,,\qquad \xi_3^2-3\xi_2\xi_4+3h\xi_1\xi_2=0\,.$$

The coordinates of the osculating plane of the curve (34) at P are given by

$$\xi_1=x^3+\cdots\,,\qquad \xi_2=-3x^2-24x^5+\cdots\,,$$
$$\xi_3=3x+15x^4+21cx^5+\cdots\,,\qquad \xi_4=-1-21ax^5+\cdots\,.$$

Prove that every five-point cubic is also a five-plane cubic of C at P, and that the five-point cubic for which $h=2$ is a six-plane cubic.

FUBINI and ČECH, 1926. 1, p. 42

10. The osculating linear complex of the osculating twisted cubic at a point P of a curve C in S_3 is the same as the osculating linear complex of C at P.

11. Defining *an anharmonic plane curve* to be a curve for which, in the notation of Exercise 4,

$$\theta_8^3/\theta_3^8=\text{const.},$$

use the Halphen canonical form of equation (5) mentioned in Exercise 4 to show that an anharmonic curve admits a one-parameter group of projective transformations into itself. Show that for each anharmonic curve there exists a triangle such that the cross ratio of a point P on the curve and the three points where the tangent at P meets the triangle is constant. WILCZYNSKI, 1906. 1, p. 86

12. At a coincidence point of a plane curve the osculating cubic of the curve has a node. One of the branches of the cubic has eight-point contact with the curve at the point, and the other branch merely passes through the point.

HALPHEN, 1918. 4, p. 207 and p. 205

13. If a curve C in ordinary space belongs to a linear complex, the tangent line at a variable point of C intersects the osculating plane at a fixed point P of C in a point whose locus is a plane curve with a sextactic point at P.

HALPHEN, 1880. 1; 1918. 4, pp. 430–31

14. In the notation of equations (34), the equation of the cubic cone having its vertex at a point P of a curve C, having the tangent line of C at P for double generator, and having eleven-point contact with C at P, is

$$xyz-y^3-z^3+c(y^2-zx)z=0 .$$

This cone has twelve-point contact in case $d-2a-c^2=0$.

HALPHEN, 1880. 1; 1918. 4, p. 432

15. In a plane consider an integral curve C of equation (5) and define a general one of the three line coordinates ξ of the tangent at a point x of C by the formula

$$\xi=e^{3\int p_1 dt}\,(x,\,x') .$$

Then show that ξ satisfies the Lagrange adjoint of (5), namely

$$\xi'''-3p_1\xi''+3(p_2-2p_1')\xi'-(p_3-3p_2'+3p_1'')\xi=0 .$$

Generalize this result for a curve in space S_n.

16. In ordinary space two non-singular quadric surfaces having one, and only one, generator in common intersect elsewhere in a twisted cubic.

17. Prove that the differential equation of a coincidence curve can be reduced to the form $x'''+x=0$. Hence show that every coincidence curve is projectively equivalent to a logarithmic spiral which intersects all of its radii at an angle of 30°.

WILCZYNSKI, 1906. 1, p. 69

18. If two plane curves have at a point P contact of order $k-1$, and if non-homogeneous projective coordinates are chosen so that the expansions representing these two curves in the neighborhood of P are, respectively, $y=ax^k+\cdots$, $y=bx^k+\cdots$ $(a\neq b)$, then the ratio a/b is a projective invariant of the curves, with the following interpretation. Consider a transversal straight line cutting the two curves in points R, S, near P, and cutting the common tangent at P in a point T, also near P. Let M be any fourth point on the transversal. Then the limit, as T approaches P, of the cross ratio $(RSTM)$ is a/b, provided that the limit of the transversal is not the common tangent, and also provided that the limit of the point M is not the point P.

SEGRE, 1897. 1, p. 170

19. In Exercise 18 let $k=2$, and calculate the cross ratio $(RSTM)$ as a power series in the coordinate not zero of the point T. The term of order zero is a/b and is independent of the limit of the transversal and of the point M. The term of the first order depends on the limit of the transversal but not on the limit of M. This term vanishes if the limit of the transversal is the harmonic conjugate of the common tangent with respect to the two lines projecting from P the two points of intersection of an arbitrary conic having four-point contact with one curve at P and another conic having similar contact with the other curve. The term of the second order

depends on the limit of the point M, when the limit of the transversal is fixed in the way just described. The point in which this limit line touches the curve enveloped by the transversal is projectively related to the point M that makes the term of the second order vanish. The double points of the projectivity thus determined on the limit line, one of which is P, and the two points in which this line meets the osculating conics of the two curves at P, have a cross ratio which is a/b. Discuss the situation when $a=b$. BOMPIANI, 1926. 8

20. If two curves in ordinary space have at a point P contact of order n, and if their principal plane is different from their common osculating plane at P, there exists a line through P in the principal plane such that the projections of the two curves from any point on this line onto any plane have contact of order $n+2$. On this line, called *the principal line*, there is a point, called *the principal point*, such that the projections of the curves from this point have contact of order $n+3$. Applying this result to a curve and its osculating twisted cubic, let $\beta=0$ and calculate the expansion (41) to terms of the eighth order; thus show that the equations of the principal line are $\beta=2\alpha+a\gamma=0$, and that the coordinates of the principal point are

$$[-a/2(b-ac),\ 0,\ 1/(b-ac)]\,.$$

BOMPIANI, 1926. 9

CHAPTER II

RULED SURFACES

Introduction. Just as a curve can be thought of as the path of a moving point, so a ruled surface can be regarded as the locus of a moving straight line. A ruled surface can also be described as a one-parameter family, or single infinity, of straight lines, which are called *the generators* of the surface. *The essential characteristic property of a ruled surface is that through each point of the surface there passes at least one straight line that lies entirely on the surface.* A quadric cone is an example of a ruled surface; and a non-singular quadric surface in ordinary space is an example, in fact the only example, of a surface that is ruled in two ways so that there are precisely two generators through every point of the surface.

In order to avoid needless repetition it seems desirable to present in the opening section of this chapter the exact definition of *an analytic surface*, and to develop the elements of the projective theory of such surfaces. The next section is devoted to the special class of ruled surfaces called *developable surfaces.* The foundations are laid in the following section for the projective theory of *general ruled surfaces* in a linear space of n dimensions.

Wilczynski in 1906 published in book form his projective theory of curves and of ruled surfaces in ordinary space, to which during the preceding five years he had devoted about ten memoirs, published mostly in the *Transactions of the American Mathematical Society.* This theory was a significant contribution to projective differential geometry. With it Wilczynski gave to the world a new method in geometry and established himself as the founder of a new school of geometers.

Wilczynski's theory of ruled surfaces is based upon a consideration of the invariants and covariants of a system of two ordinary linear homogeneous differential equations of the second order in two dependent variables, under a suitably chosen group of transformations. We shall employ this analytic basis in four sections devoted to *ruled surfaces in ordinary space*, namely, Sections 11–14. In these we shall not elaborate the theory of complete systems of invariants and covariants. However, we shall preserve Wilczynski's notations and include some of his most fundamental geometrical results.

In the concluding section of this chapter a brief account is given of one of Bompiani's contributions to the theory of ruled surfaces in hyperspace.

This is the theory of *quasi-asymptotic* curves, which are generalizations of ordinary asymptotic curves.

8. Elements of the theory of analytic surfaces. The purpose of this section is to develop some of the elementary parts of the theory of general analytic surfaces, which will be used later on in discussing ruled surfaces. *Parametric curves* and *curvilinear coordinates* on a surface, and curvilinear representations of *families* and *nets* of curves on a surface, are introduced. Consideration of the tangent line and the osculating plane at a point of a curve on a surface leads to the definition of *the tangent plane* at a point of a surface, and to the definition of *asymptotic curves.*

A surface can be described as a two-parameter family of points. More precisely, an analytic surface may be defined as follows. *If the $n+1$ homogeneous coordinates x of a point P_x in a linear space of n dimensions S_n are given as single-valued analytic functions of two independent variables u, v, by equations of the form*

$$x = x(u, v) \ , \tag{1}$$

then the locus of P_x as u, v vary is an analytic surface S. Equation (1) is spoken of as *the parametric vector equation* of the surface S. If the coordinates x satisfy no linear homogeneous partial differential equation of the first order of the form

$$ax_u + bx_v + cx = 0 \ , \tag{2}$$

in which subscripts indicate differentiation, and the coefficients a, b, c are scalar functions of u, v not all zero, then the surface S is called a *proper* analytic surface. In fact, if the coordinates x of a point P_x are given as analytic functions of u, v and are supposed to satisfy an equation of the form (2), then integration of this equation would show that these coordinates could be expressed by the formula

$$x_i = \varphi f_i(t) \qquad (i=1, \ldots, n+1) \ ,$$

wherein φ is a function of u, v which is a particular solution of equation (2), and t is a function of u, v which is a particular solution of the equation $at_u + bt_v = 0$, while the f_i are functions of t, as is indicated by the notation. Then, since the ratios of the coordinates x_i would be functions of the single variable t, the locus of the point P_x would reduce to a curve, which might further degenerate into a single fixed point (see Ex. 1 of Chap. III). In what follows, when we speak of *a surface* this will be understood to be *a proper analytic surface* unless the contrary is indicated. Moreover, singular points

where equation (2) may happen to be satisfied conditionally on such a surface will be avoided.

If the parameter v is fixed while u varies, the locus of the point P_x is a curve, called a *u-curve*, and sometimes denoted by C_u, on a surface S. There are ∞^1 u-curves on S, along each of which $v=$ const. and therefore $dv=0$. Similarly there are ∞^1 *v-curves* on the surface S, along each of which $u=$ const. and $du=0$. These curves are called *the parametric curves* on the surface S, and their tangents are called respectively *u-tangents* and *v-tangents*, and collectively *the parametric tangents. Two one-parameter families of curves on a surface S are said to form a net in case through each point of S there passes just one curve of each family, the two tangents of the curves at the point being distinct.* We shall suppose from now on that, on the portion of the surface S under consideration, the parametric curves form a net, which will be called *the parametric net.*

Each pair of values of the parameters u, v locates a point P_x on a surface S. In a region on S where points P_x and pairs of values of u, v are in continuous one-to-one correspondence, u, v are correctly called *coordinates* of P_x. It is customary to designate these coordinates as *curvilinear coordinates* to distinguish them from the projective homogeneous coordinates x.

Any curve on a surface S can be represented by a *curvilinear equation*, i.e., an equation in curvilinear coordinates, of the form $\varphi(u, v)=0$, and can also be represented parametrically by placing

$$u=u(t)\ , \qquad v=v(t)\ . \tag{3}$$

Any ∞^1 curves, except the curves $u=$ const., on the surface S can be represented by a *curvilinear differential equation* of the form

$$dv-\lambda(u, v)du=0\ ; \tag{4}$$

and any net of curves on S has a curvilinear differential equation of the form

$$Adu^2+2Bdudv+Cdv^2=0 \qquad (AC-B^2\neq 0)\ . \tag{5}$$

In particular, *the differential equation of the parametric net is* $dudv=0$.

It will now be proved that *at an ordinary point on a surface the tangent lines of all the curves on the surface through the point form a flat pencil with its center at the point.* The tangent line at a point P_x of a curve C with equations (3) on a surface S is determined, as in Figure 4, by P_x and the point x', where

$$x'=x_u u'+x_v v'\ , \tag{6}$$

accents indicating differentiation with respect to t. If the point P_x remains fixed while the curve C through P_x varies in all possible ways on the surface S, then x_u, x_v remain fixed while the derivative dv/du varies. The locus of the point x' is therefore the line joining the points x_u, x_v, the first of which lies on the u-tangent of S at P_x, and the second on the v-tangent. Therefore the locus of the tangent line xx' is a flat pencil lying in the plane of the points x, x_u, x_v and with its center at the point P_x, as was to be proved.

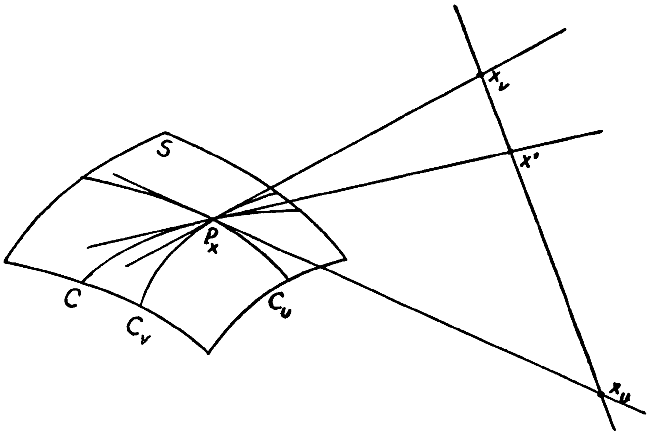

FIG. 4

We are now ready to state the classical definition of the tangent plane. *The tangent plane at a point of a surface is the plane containing the tangent lines at the point of all the curves on the surface through the point.* The tangent plane at a point x of a surface S referred to parameters u, v is determined by the points x, x_u, x_v. If the surface S is in ordinary space S_3, the equation of the tangent plane can be written in the determinantal form

$$(X,\ x,\ x_u,\ x_v)=0\ , \tag{7}$$

in which the point X is variable on the plane.

Since the position of the tangent line at a point P_x of a curve C on a surface S is determined by the derivative dv/du calculated from the curvilinear representation of C, this derivative is called *the direction* of the curve C at the point P_x.

An asymptotic curve on a surface may be defined projectively as follows. *A curve on a surface is an asymptotic curve in case at each of its points its osculating plane coincides with the tangent plane of the surface at the point.*

The osculating plane at a point P_x of a curve (3) on a surface S is determined by x, x', x'', where x' is given by equation (6) and x'' by

$$x''=x_{uu}u'^2+2x_{uv}u'v'+x_{vv}v'^2+x_uu''+x_vv'' . \tag{8}$$

If the surface S is in ordinary space, the curve (3) is an asymptotic curve in case not only x, x' but also x'' satisfy equation (7) when substituted therein in place of X. Substitution of x'' for X in (7), and reduction of the result, lead to the following theorem.

The curvilinear differential equation of the asymptotic curves on a surface (1) *in ordinary space is*

$$Ldu^2+2Mdudv+Ndv^2=0 , \tag{9}$$

where the coefficients L, M, N are the determinants of the fourth order defined by

$$L=(x_{uu},\ x,\ x_u,\ x_v) , \qquad M=(x_{uv},\ x,\ x_u,\ x_v) , \qquad N=(x_{vv},\ x,\ x_u,\ x_v) . \tag{10}$$

We conclude this section with a few remarks pertaining to asymptotic curves. Usually the asymptotic curves on a surface in ordinary space form a net. The exceptional case will be considered at the close of the next section.

The tangent line at a point of a straight line is the line itself; the osculating plane at a point of a straight line is indeterminate. Every straight line on a surface is included among the asymptotic curves on the surface, since the tangent plane of the surface at a point of the straight line is an osculating plane of the straight line.

The reason why the asymptotic curves on a surface are called by this name is a historical one. They were first defined* as those curves whose tangents at each point of the surface are the asymptotes of the conic now commonly called *the Dupin indicatrix* of the surface at the point. This property is, however, not a projective property.

9. Developable surfaces. The reason why developable surfaces are so named is that *in ordinary space analytic developable surfaces are the only analytic surfaces that can be developed upon, or applied to, a plane.* This means that it is possible to set up a one-to-one correspondence between the points of an analytic developable surface in ordinary space and the points of a plane such that corresponding curves between corresponding points have equal lengths, and no other analytic surfaces in ordinary space have this property. Since this definition is not of a projective nature, it is neces-

* Dupin, 1813. 1, p. 51.

sary to substitute for it a purely projective definition of developable surfaces before admitting this important class of ruled surfaces to the domain of projective differential geometry. We proceed to do so in this section. We then set up the parametric vector equation of a developable surface and study some of the fundamental properties of developable surfaces.

In space S_n *a developable surface is defined to be the locus of the tangent lines of a curve*, and is sometimes called simply a *developable*. The tangents are known as *the generators* of the developable; the curve is called *the cuspidal edge*, or *edge of regression*, of the developable; and the point of contact of a generator with the cuspidal edge is designated as *the focal point* of the generator.

Certain special cases should be mentioned here. If the edge of regression is a plane curve, the developable is obviously all or part of the plane of the curve. If the edge is a straight line, the developable evidently reduces to this line and is not a proper surface. If the edge reduces to a fixed point, a little meditation shows that the developable is to be thought of as a cone with its vertex at the point.

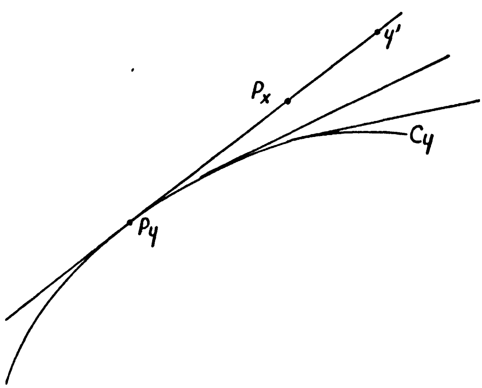

Fig. 5

The next problem is to write the parametric vector equation of a developable. Let the parametric vector equation of the edge of regression C_y of a developable be $y = y(t)$, as indicated in Figure 5. This curve is the locus of a point P_y; and any point P_x, except P_y, on the tangent of C_y at P_y is defined by placing

$$x = y' + uy \qquad (u \text{ scalar}) \; . \tag{11}$$

If the parameter t is fixed while u varies, the locus of the point P_x is the tangent line; but if u, t both vary, the locus of P_x is the developable surface. Hence we reach the conclusion:

Equation (11) *is the parametric vector equation of a developable surface.*

Certain properties of developable surfaces merit attention. The tangent plane at a point P_x of the developable (11) is determined by the points x, x_u, x_t. If the derivatives x_u, x_t are calculated it is easy to see that the tangent plane may also be regarded as determined by the points y, y', y''. But these

points determine the osculating plane of the edge of regression at the point P_y. Thus we prove the theorem:

The tangent plane at every point of a generator of a developable surface is the same plane, and is the osculating plane of the edge of regression of the developable at the focal point of the generator.

It follows from this theorem that *a developable surface has only* ∞^1 *tangent planes instead of the usual* ∞^2. Conversely, in ordinary space the planes of any analytic one-parameter family osculate a curve, according to the conclusion of Section 3, and therefore envelop a developable surface.

Using the expression for x given in equation (11), let us replace v by t in equations (9), (10), and calculate the determinants L, M, N defined by (10). We find $L=M=0$, $N=(y''', y'', y', y)$. If $N=0$, the curve C_y is a plane curve, and conversely. Let us suppose $N\neq 0$. Then (9) shows that the *asymptotic curves on a developable surface immersed in ordinary space consist only of the generators of the surface.* Consequently these curves do not form a net. Conversely, it can be shown that *if the asymptotic curves on a surface S immersed in ordinary space form only a one-parameter family, then S is a developable surface.* For, in the first place, we have $LN-M^2=0$. If the family of asymptotic curves is taken for the u-curves on the surface S, then $L=M=0$, and consequently x satisfies two equations of the form

$$x_{uu}=px+\alpha x_u+\beta x_v\ , \qquad x_{uv}=cx+ax_u+bx_v\ , \tag{12}$$

whose coefficients are scalar functions of u, v. The coordinates ξ of the tangent plane (7) can be defined by

$$\xi=(x,\ x_u,\ x_v)\ ;$$

and in the presence of equations (12) differentiation yields $\xi_u=(\alpha+b)\xi$. Therefore the tangent plane at every point of a u-curve is the same plane, the ratios of the coordinates ξ being independent of u. Consequently the surface S is developable. Thus the following theorem is proved.

The asymptotic curves on a surface immersed in ordinary space fail to form a net if, and only if, the surface is developable.

10. Foundations of the theory of ruled surfaces in S_n. The parametric vector equation of a general ruled surface in a linear space of n dimensions can be written very simply by the aid of two curves with their points in one-to-one correspondence. After writing this equation and deducing the most elementary properties appertaining to the tangent planes of a ruled surface, the remainder of this section will be taken up with an introduction

to Bompiani's theory of the various osculating linear spaces that can be associated with an element of a curve on a ruled surface.

A ruled surface is a one-parameter family of straight lines. Therefore *every developable surface is ruled.* But not every ruled surface is developable. For example, among the non-developable ruled surfaces in ordinary space is the quadric $x_2x_3 - x_1x_4 = 0$, which may also be defined by the parametric equations

$$x_1 = 1 , \qquad x_2 = u , \qquad x_3 = v , \qquad x_4 = uv , \tag{13}$$

and which has on it the two *distinct* one-parameter families of straight lines $x_2 = ux_1$, $x_4 = ux_3$ and $x_3 = vx_1$, $x_4 = vx_2$. These lines can be shown by equations (9), (10) to be the asymptotic curves on the quadric, and they constitute the parametric net for the representation used in equations (13).

In order to write the parametric vector equation of a general ruled surface, let us consider in space S_n two curves C_y, C_z as in Figure 6. Let us

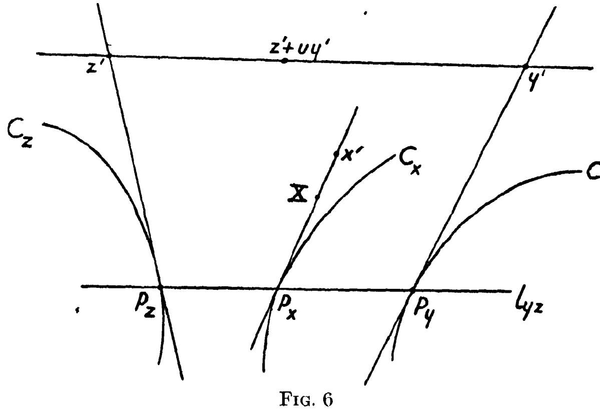

FIG. 6

refer these curves to the same parameter t, and let us join a pair of points P_y, P_z (supposed distinct) corresponding to the same value of t by a straight line l_{yz}. The locus of l_{yz}, as t varies, is a ruled surface R, and every ruled surface can be generated in this way. Any point P_x, except P_y, on l_{yz} is defined by placing

$$x = z + uy \qquad (u \text{ scalar}) . \tag{14}$$

If t is fixed while u varies, the locus of the point P_x is a generator of the ruled surface R. If u, t both vary, the locus of P_x is the surface R. Hence the useful result:

Equation (14) *is the parametric vector equation of a general ruled surface.*

It may be observed that the ruled surface R is developable if $z=y'$, and is a cone if $z=\text{const}$. The case that the surface reduces to a single fixed line is excluded. The curves C_y, C_z will be called *director curves* of the ruled surface R. It is clear that any two distinct curves on a ruled surface may be taken as the director curves.

We shall next show that *if there are two distinct curves on a ruled surface such that their tangent lines at the points where they cross each generator are coplanar, then the ruled surface is developable.* For, if the two curves are taken as the director curves C_y, C_z, then the points y, z, y', z' are coplanar and there exists a relation of the form

$$ay+bz+cy'+dz'=0 , \tag{15}$$

where a, b, c, d are scalar functions of t. This equation is equivalent to

$$(cy+dz)'=(c'-a)y+(d'-b)z .$$

Hence, when t varies, the point $cy+dz$ describes a curve to which the generator l_{yz} is tangent. Therefore all the generators of the surface are the tangents of a curve, and the surface is developable, as was to be proved.

The tangent plane at a point P_x of a ruled surface (14) is determined by x, x_u, x_t and therefore by y, z, $z'+uy'$. Consequently *the tangent plane at a point of a ruled surface contains the generator of the surface through the point*, as is also evident geometrically. Conversely, *every plane through a generator of a non-developable ruled surface is tangent to the surface at just one point of the generator.* For, the plane can be regarded as determined by the points y, z, and some point $z'+ky'$ because the points y, z, y', z' are not coplanar; the tangent plane at the point $z+uy$ coincides with this plane if, and only if, $u=k$. We see, then, that the *tangent planes of a non-developable ruled surface may be assembled into a single infinity of axial pencils*, whose axes are the generators of the surface.

The cross ratio of four points P_x given by equation (14) with a fixed value of t and four distinct values of u is equal to the cross ratio of these values of u, which is also equal to the cross ratio of the corresponding four points $z'+uy'$. If the ruled surface R is not developable, this cross ratio is further equal to the cross ratio of the tangent planes of the surface at the first four points. So we have *the correlation* of Chasles:*

The cross ratio of four points on a generator of a non-developable ruled surface is equal to the cross ratio of the four planes tangent to the surface at these points.

* Chasles, 1839. 1, p. 53.

Any curve C_x on a ruled surface (see Fig. 6) in a linear space S_n, except the curve C_y and the generators, can be defined by placing $u=u(t)$ in equation (14). The osculating linear space S_k at a point P_x of the curve C_x is determined by the points $x, x', \ldots, x^{(k)}$, where

$$
(16) \qquad \begin{cases} x'=z'+uy'+u'y\ , \\ x''=z''+uy''+2u'y'+u''y\ , \\ \cdots \quad \cdots \\ x^{(k)}=z^{(k)}+\sum_{i=0}^{k} C_{k,i}\, u^{(i)}y^{(k-i)}\ . \end{cases}
$$

The linear space of least dimensions that contains a set of linear spaces may be called briefly *the ambient* thereof. So the ambient, or joining space, of two distinct points is the line joining them; and the ambient of a point and a line not in united position is the plane determined by them.

We are now ready to state an important definition. *The osculating* space $S(k, r)$ with respect to an element E_r of a curve at a point P_x of a ruled surface R is the ambient of the osculating space S_k at the point P_x of every curve on the surface R that passes through P_x and has at P_x the same element E_r* $(r<k)$. For example, the osculating space $S(2, 1)$ with respect to an element E_1 of a curve at a point P_x of a ruled surface R is the ambient of the osculating plane at the point P_x of every curve on the surface R through P_x having at P_x the same tangent line. Sometimes we say that the space $S(2, 1)$ is *in the direction of* the curve. In particular, when $r=0$, our definition becomes the following statement. *The osculating space $S(k, 0)$ at a point P_x of a ruled surface R is the ambient of the osculating space S_k at the point P_x of every curve on the surface R through P_x.* For example, the osculating, or tangent, space $S(1, 0)$ at a point P_x of a ruled surface R is the ambient of the tangent line at the point P_x of every curve on the surface R through P_x; so *the space $S(1, 0)$ is the tangent plane* at the point P_x, of the surface R. It may be remarked that in the definition of the space $S(k, r)$ the ruled surface could be replaced by a general surface or variety (see § 52, Chap. VII).

It is a fundamental problem to determine the dimensionality of the space $S(k, r)$ just defined. Every curve on a ruled surface R, through a point P_x of R and having at P_x the same element E_r, has at P_x the same values of $u, u', \ldots, u^{(r)}$; but two such curves usually have different values of $u^{(r+1)}, \ldots, u^{(k)}$. An upper limit on the dimensionality of the space $S(k, r)$, which is ordinarily the actual dimensionality, can be determined by holding $u, u', \ldots, u^{(r)}$ fixed while $u^{(r+1)}, \ldots, u^{(k)}$ vary in equations

* Bompiani, 1914. 2, p. 307.

(16), and thus determining the greatest number of linearly independent points in the space $S(k, r)$. In this process $x, x', \ldots, x^{(r)}$ are fixed. But $u^{(r+1)}$ appears in the formula for $x^{(r+1)}$ as the coefficient of y, so that the point y can be taken as one of the points sought; in the presence of the point y, the point x can be replaced by the point z. Similarly, $u^{(r+1)}, u^{(r+2)}$ appear in the formula for $x^{(r+2)}$ as the coefficients of y', y respectively, so that the point y' can be taken as one of the points sought; in the presence of the points, y, y' the point x' can be replaced by the point z'. Continuing this reasoning we come to the last stage where we see that $u^{(r+1)}, \ldots, u^{(k)}$ appear in the formula for $x^{(k)}$ as the coefficients of $y^{(k-r-1)}, \ldots, y', y$ respectively, so that the point $y^{(k-r-1)}$ can be taken as one of the points sought; in the presence of the points $y, y', \ldots, y^{(k-r-1)}$ the point $x^{(k-r-1)}$ can be replaced by the point $z^{(k-r-1)}$. Then, in the presence of y, z and their first $k-r-1$ derivatives, the point $x^{(k-r)}$ can be replaced by the point $z^{(k-r)}+uy^{(k-r)}$; the point $x^{(k-r+1)}$ by the point $z^{(k-r+1)}+uy^{(k-r+1)}+(k-r+1)u'y^{(k-r)}$, and so on until finally the point $x^{(k)}$ is replaced by the last of the points mentioned in the following statement.

The osculating space $S(k, r)$ with respect to an element E_r of a curve at a point P_x of a ruled surface R in a space $S_n(n \geqq 2k-r)$ is determined by the $2k-r+1$ points

$$
(17) \quad \left\{
\begin{aligned}
& y, y', \ldots, y^{(k-r-1)}, \quad z, z', \ldots, z^{(k-r-1)}, \quad z^{(k-r)}+uy^{(k-r)}, \\
& z^{(k-r+1)}+uy^{(k-r+1)}+(k-r+1)u'y^{(k-r)}, \quad z^{(k-r+2)}+uy^{(k-r+2)} \\
& \qquad +(k-r+2)u'y^{(k-r+1)}+(k-r+2)(k-r+1)u''y^{(k-r)}/2, \ldots, \\
& z^{(k)}+\sum_{i=0}^{r} C_{k,i}\, u^{(i)}y^{(k-i)},
\end{aligned}
\right.
$$

and is therefore ordinarily a space S_{2k-r}.

In particular, the space $S(k, 0)$ is ordinarily the space S_{2k} of even dimensions which is determined by the $2k+1$ points

$$
(18) \qquad y, y', \ldots, y^{(k-1)}, \quad z, z', \ldots, z^{(k-1)}, \quad z^{(k)}+uy^{(k)} .
$$

Then the space $S(2, 0)$ is the space S_4 of the points $y, y', z, z', z''+uy''$; as we have already seen, the space $S(1, 0)$ is the tangent plane, which is determined by the points $y, z, z'+uy'$.

A certain osculating linear space along a generator, rather than at a point, of a ruled surface will now be defined. The ambient of the spaces $S(k, 0)$ at all points of a generator l of a ruled surface R is studied by allowing u to

vary in formulas (18). This ambient is thus seen to be determined by the $2k+2$ points

$$y, y', \ldots, y^{(k)}, \qquad z, z', \ldots z^{(k)}, \tag{19}$$

and is therefore ordinarily a space S_{2k+1}. This space of odd dimensions is called *the osculating space S_{2k+1} along the generator l of the ruled surface R. This is in fact the linear space of least dimensions that contains the osculating spaces S_k at points of l of all curves on R that intersect l.* In particular, the osculating space S_5 along a generator l of a ruled surface R is seen to be the ambient of the osculating planes at points of l of all curves on R that intersect l. Finally, the osculating, or tangent, ordinary space S_3 along a generator l of a ruled surface R is the ambient of the tangent planes of R at all the points of l, and is also the ambient of the tangent lines at points of the generator l of all curves on the surface R that intersect l. Sometimes this space S_3 is thought of as determined by the generator l and a consecutive generator.

11. The differential equations of a ruled surface in S_3. Wilczynski's system of differential equations, defining a ruled surface in ordinary space S_3 except for a projective transformation, is fundamental in this and the next three sections. In this section canonical forms are obtained for the differential equations by means of certain transformations of the independent and dependent variables which do not disturb the surface.

Let us consider* a non-developable ruled surface R with parametric vector equation (14), immersed in ordinary space S_3. The determinant (y, z, y', z') does not vanish identically, because y, z, satisfy no relation of the form (15). Consequently it can be shown that y, z satisfy a system of differential equations of the form

$$\begin{cases} y''+p_{11}y'+p_{12}z'+q_{11}y+q_{12}z=0\,, \\ z''+p_{21}y'+p_{22}z'+q_{21}y+q_{22}z=0\,, \end{cases} \tag{20}$$

in which the coefficients are scalar functions of the parameter t, and accents indicate differentiation with respect to t. In fact, if the four pairs of coordinates y_i, z_i $(i=1, \ldots, 4)$ are substituted in turn in the first of equations (20) the resulting four linear algebraic equations can be solved uniquely for the coefficients $p_{11}, p_{12}, q_{11}, q_{12}$; and the other coefficients can be determined similarly. Then R is called an *integral ruled surface* of system (20), and every projective transform of R is also an integral ruled surface of (20). Conversely, when equations (20) are given, the theory of differential equa-

* Wilczynski, 1906. 1, Chaps. IV, V, VI.

tions tells us that these equations possess four pairs of solutions y_i, z_i $(i=1, \ldots, 4)$ forming a fundamental set. When these solutions are interpreted as the coordinates of two points P_y, P_z, we see that *equations* (20) *define a ruled surface in space* S_3 *except for a projective transformation.*

Let us consider *the transformation of dependent variables,*

$$y=\alpha\eta+\beta\zeta\ , \qquad z=\gamma\eta+\delta\zeta \qquad (\Delta=\alpha\delta-\beta\gamma\neq 0)\ , \tag{21}$$

in which the coefficients α, β, γ, δ are scalar functions of t. The effect of this transformation on equations (20) is to produce the system of equations

$$\left\{\begin{aligned} &\alpha\eta''+\beta\zeta''+(2\alpha'+p_{11}\alpha+p_{12}\gamma)\eta'+(2\beta'+p_{11}\beta+p_{12}\delta)\zeta' \\ &\quad+(p_{11}\alpha'+p_{12}\gamma'+\alpha''+q_{11}\alpha+q_{12}\gamma)\eta+(p_{11}\beta'+p_{12}\delta'+\beta''+q_{11}\beta+q_{12}\delta)\zeta \\ &\gamma\eta''+\delta\zeta''+(2\gamma'+p_{21}\alpha+p_{22}\gamma)\eta'+(2\delta'+p_{21}\beta+p_{22}\delta)\zeta' \\ &\quad+(p_{21}\alpha'+p_{22}\gamma'+\gamma''+q_{21}\alpha+q_{22}\gamma)\eta+(p_{21}\beta'+p_{22}\delta'+\delta''+q_{21}\beta+q_{22}\delta)\zeta \end{aligned}\right. \tag{22}$$

Geometrically, the transformation (21) is a change of director curves on an integral ruled surface from C_y, C_z to C_η, C_ζ, leaving the surface unchanged. Therefore *systems* (20) *and* (22) *have the same integral ruled surfaces.*

The transformation (21) can be used to simplify the fundamental differential equations. If α, γ and β, δ are two pairs of solutions of the equations

$$2\rho'+p_{11}\rho+p_{12}\sigma=0\ , \qquad 2\sigma'+p_{21}\rho+p_{22}\sigma=0 \tag{23}$$

for ρ, σ, then equations (22) become

$$\left\{\begin{aligned} 4(\alpha\eta''+\beta\zeta'')&=(\alpha u_{11}+\gamma u_{12})\eta+(\beta u_{11}+\delta u_{12})\zeta\ , \\ 4(\gamma\eta''+\delta\zeta'')&=(\alpha u_{21}+\gamma u_{22})\eta+(\beta u_{21}+\delta u_{22})\zeta\ , \end{aligned}\right. \tag{24}$$

wherein the functions u_{ik} are defined by the formulas

$$\left\{\begin{aligned} u_{11}&=2p'_{11}-4q_{11}+p_{11}^2+p_{12}p_{21}\ , \qquad & u_{12}&=2p'_{12}-4q_{12}+p_{12}(p_{11}+p_{22})\ , \\ u_{22}&=2p'_{22}-4q_{22}+p_{22}^2+p_{21}p_{12}\ , \qquad & u_{21}&=2p'_{21}-4q_{21}+p_{21}(p_{22}+p_{11})\ . \end{aligned}\right. \tag{25}$$

If we were to solve equations (24) for η'', ζ'', we would obtain two equations of the same form as (20) but with the terms in η', ζ' missing. Therefore we have the theorem:

It is possible by means of the transformation (21) *to reduce system* (20) *to a canonical form for which* $p_{ik}=0$ $(i, k=1, 2)$.

We next deduce a second canonical form of system (20). If equations

(22) are solved for η'', ζ'' the result is another system of the same form as (20) whose coefficients π_{ik}, χ_{ik} are given by the equations

$$(26)\quad \begin{cases} \Delta\pi_{11}=2(\alpha'\delta-\gamma'\beta)+p_{11}\alpha\delta+p_{12}\gamma\delta-p_{21}\alpha\beta-p_{22}\beta\gamma\ , \\ \Delta\pi_{12}=2(\beta'\delta-\delta'\beta)+p_{11}\beta\delta+p_{12}\delta^2-p_{21}\beta^2-p_{22}\beta\delta\ , \\ \Delta\chi_{11}=\alpha''\delta-\gamma''\beta+p_{11}\alpha'\delta+p_{12}\gamma'\delta-p_{21}\alpha'\beta-p_{22}\gamma'\beta+q_{11}\alpha\delta+q_{12}\gamma\delta \\ \qquad\qquad -q_{21}\alpha\beta-q_{22}\beta\gamma\ , \\ \Delta\chi_{12}=\beta''\delta-\delta''\beta+p_{11}\beta'\delta+p_{12}\delta'\delta-p_{21}\beta'\beta-p_{22}\delta'\beta+q_{11}\beta\delta+q_{12}\delta^2 \\ \qquad\qquad -q_{21}\beta^2-q_{22}\beta\delta\ , \end{cases}$$

and the equations obtainable therefrom by the substitution

$$(27)\qquad \begin{pmatrix} y & 1 & \eta & \alpha & \beta & a & \rho \\ z & 2 & \zeta & \delta & \gamma & b & \sigma \end{pmatrix},$$

in which a, b, ρ, σ will be defined later in equations (36) and (41); this substitution leaves all the systems of equations in this section invariant. By means of these results it can be shown, although we shall omit the calculations, that *the effect of the transformation* (21) *on the functions* u_{ik} *defined by the formulas* (25) *is given by*

$$(28)\quad \begin{cases} \Delta\bar{u}_{11}=\alpha\delta u_{11}+\gamma\delta u_{12}-\alpha\beta u_{21}-\beta\gamma u_{22}\ , \\ \Delta\bar{u}_{12}=\beta\delta u_{11}+\delta^2 u_{12}-\beta^2 u_{21}-\beta\delta u_{22}\ , \\ \Delta\bar{u}_{21}=\gamma\alpha u_{22}+\alpha^2 u_{21}-\gamma^2 u_{12}-\gamma\alpha u_{11}\ , \\ \Delta\bar{u}_{22}=\alpha\delta u_{22}+\alpha\beta u_{21}-\gamma\delta u_{12}-\beta\gamma u_{11}\ , \end{cases}$$

wherein the dash indicates the transformed functions. If the equation

$$(29)\qquad u_{12}\lambda^2+(u_{11}-u_{22})\lambda-u_{21}=0$$

has distinct roots so that its discriminant θ_4, defined by the formula

$$(30)\qquad \theta_4=(u_{11}-u_{22})^2+4u_{12}u_{21}\ ,$$

is not zero, let us take one of these roots for γ/α and the other for δ/β. Then we have $\bar{u}_{12}=\bar{u}_{21}=0$. So we reach the conclusion:

It is possible by means of the transformation (21) *to reduce system* (20) *to a canonical form for which* $u_{12}=u_{21}=0$, *provided* $\theta_4\neq 0$.

The most general transformation (21) preserving the canonical form of system (20) characterized by the conditions $u_{12}=u_{21}=0$ is subject to the

restrictions $\beta\delta=\gamma\alpha=0$ when $u_{11}-u_{22}\neq 0$. It is possible to make any transformation (21) for which $\beta=\gamma=0$, $\alpha\delta\neq 0$ without disturbing this canonical form. When the differential equations are in this canonical form, the director curves are definitely chosen, except for a mere interchange; their geometrical description will be given in Section 13.

Let us next consider *the transformation of parameter*,

$$\tau=\tau(t) \qquad (\tau'\neq 0) \ . \tag{31}$$

The effect of this transformation on system (20) is to produce another system of the same form whose coefficients, indicated by dashes, are given by the formulas

$$\begin{cases} \tau'\bar{p}_{ii}=p_{ii}+\omega\ , & \tau'\bar{p}_{ik}=p_{ik}\ , \\ \tau'^2\bar{q}_{ii}=q_{ii}\ , & \tau'^2\bar{q}_{ik}=q_{ik} \end{cases} \qquad (\omega=\tau''/\tau';\ i,\ k=1,\ 2;\ i\neq k)\ . \tag{32}$$

The effect of the transformation (31) on the functions u_{ik} is given by

$$\tau'^2\bar{u}_{ii}=u_{ii}+2\omega'-\omega^2\ , \qquad \tau'^2\bar{u}_{ik}=u_{ik}\ . \tag{33}$$

The function θ_4, which can be shown by equations (28) to be absolutely invariant under the transformation (21), is transformed by (31) into $\bar{\theta}_4$ according to the formula

$$\tau'^4\bar{\theta}_4=\theta_4\ , \tag{34}$$

and is therefore a relative invariant under the total transformation (21), (31). The geometrical significance of the absolutely invariant equation $\theta_4=0$ will be pointed out in Section 13.

It is possible to simplify still further the canonical form of system (20) characterized by the conditions $u_{12}=u_{21}=0$. The transformation

$$y=\alpha\eta\ , \qquad z=\delta\zeta\ , \qquad \tau=\tau(t) \qquad (\alpha\delta\tau'\neq 0) \tag{35}$$

leaves these conditions invariant and changes the coefficients p_{11}, p_{22} into $\bar{p}_{11}$, $\bar{p}_{22}$ according to the following formulas, which may be written by the aid of equations (26), (32), and the substitution (27):

$$\tau'\bar{p}_{11}=p_{11}+\omega+2\alpha'/\alpha\ , \qquad \tau'\bar{p}_{22}=p_{22}+\omega+2\delta'/\delta\ .$$

Therefore, if α, δ are solutions of the equations

$$2\alpha'+(p_{11}+\omega)\alpha=0\ , \qquad 2\delta'+(p_{22}+\omega)\delta=0$$

with τ and hence ω still arbitrary, we have $\bar{p}_{11}=\bar{p}_{22}=0$ and still have at our disposal the transformation

$$\tau=\tau(t)\ , \qquad y=a\eta/(\tau')^{1/2}\ , \qquad z=b\zeta/(\tau')^{1/2} \qquad (a,\, b=\text{consts.})\ . \tag{36}$$

By means of this transformation we can make $\bar{\theta}_4=1$ by choosing τ so that $\tau'^4=\theta_4$. Or instead we can make $\bar{u}_{11}-\bar{u}_{22}=1$ by choosing τ so that $\tau'^2=u_{11}-u_{22}$. Or as a third possibility we can make $\bar{u}_{11}+\bar{u}_{22}=0$ by choosing τ so that

$$4\omega'-2\omega^2+u_{11}+u_{22}=0\ .$$

12. The asymptotic curves and the osculating quadric. The asymptotic curves are of primary importance in the projective differential geometry of a ruled surface in ordinary space. Study of their properties leads, in this section, to the definition of a certain regulus called *the asymptotic regulus* along a generator of a ruled surface. There is a second regulus called *the osculating regulus* also associated with the generator. These two reguli lie upon the same quadric surface, called *the osculating quadric* along the generator of the surface. When the generator is allowed to vary over the surface we arrive at the definitions of *the flecnode congruence* and of *a derivative ruled surface.*

In ordinary space S_3 let us consider a non-developable ruled surface R which has the parametric vector equation (14), and which is an integral surface of system (20). Using the expression for x given in equation (14), let us replace v by t in equations (9), (10), and calculate the determinants $L,\ M,\ N,$ defined by (10). Then (9) shows that the following statement is true:

The asymptotic curves on the ruled surface R consist of the generators and the ∞^1 curves defined by the differential equation

$$2u'=p_{21}+(p_{11}-p_{22})u-p_{12}u^2 \qquad (u'=du/dt)\ . \tag{37}$$

The curves (37) are called *the curved asymptotics,* or sometimes simply *the asymptotics,* on the surface R. The director curves $C_y,\ C_z$ are asymptotic curves in case $p_{12}=p_{21}=0$; and every curve $u=\text{const.}$ is an asymptotic curve in case

$$p_{12}=p_{11}-p_{22}=p_{21}=0\ .$$

Since (37) is an equation* of Riccati, the cross ratio of four particular solutions is constant, and the theorem† of P. Serret follows:

* Goursat-Hedrick, 1917. 1, p. 12.

† Serret, 1860. 1, p. 169.

The cross ratio of the four points in which four curved asymptotics on a ruled surface in ordinary space intersect a generator is the same for all generators of the surface.

The locus of the asymptotic tangents at the points of a generator l of a ruled surface R will now be proved to be a regulus, which will be called *the asymptotic regulus* along the generator l of the surface R. It is sufficient to prove that these tangents lie on a quadric surface. Any curve C_x, except C_y and the generators (see Fig. 6), on the surface R can be defined by placing $u=u(t)$ in equation (14). The tangent line of the curve C_x, at the point P_x where C_x crosses the generator l, is determined by P_x and the point X defined by placing

$$X=x'+vx \qquad (v \text{ scalar}) .$$

If the curve C_x is an asymptotic curve we find by means of equations (14), (37) the expression

$$X=x_1y+x_2z+x_3y'+x_4z' ,$$

wherein the coefficients $x_1, \ldots, x_4$ are given by

$$\left\{\begin{array}{l} x_1=uv+[p_{21}+(p_{11}-p_{22})u-p_{12}u^2]/2 , \\ x_2=v , \qquad x_3=u , \qquad x_4=1 . \end{array}\right. \tag{38}$$

These coefficients are the local coordinates of the point X referred to the local tetrahedron y, z, y', z' with suitably chosen unit point. When u, v vary, t being fixed, equations (38) are the parametric equations of a quadric surface whose algebraic equation is found, by homogeneous elimination of u, v, to be

$$2(x_2x_3-x_1x_4)=p_{12}x_3^2-(p_{11}-p_{22})x_3x_4-p_{21}x_4^2 . \tag{39}$$

This completes the proof.

We now state the definition of the osculating regulus. *The osculating regulus along a generator l of a ruled surface R is the limit of the regulus determined by l and two neighboring generators of R as each of these independently approaches l, remaining on R.* The quadric (39) is called *the osculating quadric* along the generator l of the ruled surface R because the osculating regulus lies upon it, as we shall now demonstrate. Referring to Figure 7, let us consider a point Y near the point P_y on the director curve C_y of the ruled surface R, and consider also the corresponding point Z near P_z on C_z. Taylor's expansion gives

$$Y=y+y'\Delta t+y''\Delta t^2/2+y'''\Delta t^3/6+ \cdots ,$$
$$Z=z+z'\Delta t+z''\Delta t^2/2+z'''\Delta t^3/6+ \cdots ,$$

where Δt is the increment of t corresponding to displacement on the curve C_y from the point P_y to the point Y. Equations (20) express y'', z'' as linear

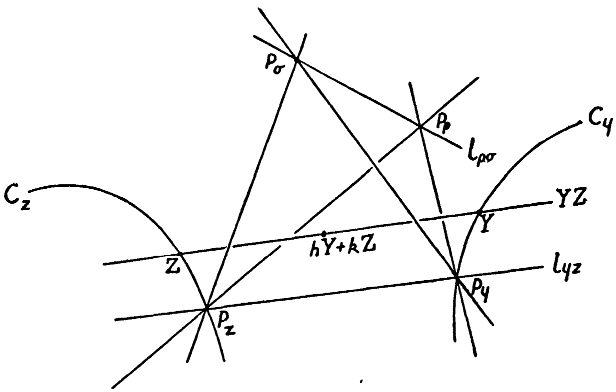

FIG. 7

combinations of y, z, y', z'; after differentiating equations (20), the third derivatives y''', z''' can be so expressed; and so on for derivatives of y, z of any required order. The point defined by the formula

$$hY+kZ \qquad (h,\ k \text{ scalars})$$

is on the generator YZ of the surface R, and we find

$$hY+kZ=x_1y+x_2z+x_3y'+x_4z'\ ,$$

where the coefficients $x_1, \ldots, x_4$ are given by

$$(40)\quad \begin{cases} x_1=h-(q_{11}h+q_{21}k)\Delta t^2/2+\cdots, \qquad x_2=k-(q_{12}h+q_{22}k)\Delta t^2/2+\cdots, \\ x_3=h\Delta t-(p_{11}h+p_{21}k)\Delta t^2/2+[h(-p'_{11}-q_{11}+p_{11}^2+p_{12}p_{21}) \\ \qquad\qquad +k(-p'_{21}-q_{21}+p_{21}p_{11}+p_{22}p_{21})]\Delta t^3/6+\cdots, \\ x_4=k\Delta t-(p_{12}h+p_{22}k)\Delta t^2/2+[h(-p'_{12}-q_{12}+p_{11}p_{12}+p_{12}p_{22}) \\ \qquad\qquad +k(-p'_{22}-q_{22}+p_{21}p_{12}+p_{22}^2)]\Delta t^3/6+\cdots. \end{cases}$$

Demanding that the general equation of a quadric surface be satisfied by the power series (40) for $x_1, \ldots, x_4$ identically in h, k and Δt as far as the terms in Δt^2, we obtain equation (39) again, thus completing the demonstration. We observe that *an ordinary tangent of a ruled surface intersects two consecutive generators*, and that *an asymptotic tangent intersects three consecutive generators* at its point of contact.

It is sometimes convenient to use the local tetrahedron (see Fig. 7) y, z, ρ, σ, the points ρ, σ being defined by placing

$$\rho=2y'+p_{11}y+p_{12}z\ ,\qquad \sigma=2z'+p_{21}y+p_{22}z\ . \tag{41}$$

If a point has coordinates $x_1, \ldots, x_4$ referred to the tetrahedron y, z, y', z', and coordinates $y_1, \ldots, y_4$ referred to the tetrahedron y, z, ρ, σ, the identity

$$p(x_1y+x_2z+x_3y'+x_4z')=y_1y+y_2z+y_3\rho+y_4\sigma\ ,$$

where p is a proportionality factor, gives the equations of transformation between the tetrahedrons, namely,

$$\begin{cases} px_1=y_1+p_{11}y_3+p_{21}y_4\ , & px_2=y_2+p_{12}y_3+p_{22}y_4\ , \\ px_3=2y_3\ , & px_4=2y_4\ . \end{cases} \tag{42}$$

We may sometimes neglect to write the factor of proportionality when its presence, as in these formulas, is not essential.

Referred to the tetrahedron y, z, ρ, σ the equation of the osculating quadric becomes

$$y_2y_3-y_1y_4=0\ . \tag{43}$$

This equation can be used to show that *the simple skew quadrilateral* $y\rho\sigma z$ *lies entirely on the osculating quadric.* That the lines $y\rho$ and $z\sigma$ do so is evident when one observes that the local equations of these lines are $y_2=y_4=0$ and $y_1=y_3=0$ respectively. The line yz or $y_3=y_4=0$ of course lies on the quadric, and so also does the line $\rho\sigma$ or $y_1=y_2=0$. It follows that *the lines* $y\rho$ *and* $z\sigma$ *are asymptotic tangents, and the line* $\rho\sigma$ *is a generator of the osculating regulus.*

We propose to study the line $\rho\sigma$ a little more closely. The effect of the transformation (31) on ρ, σ is found by the formulas (32) to be given by

$$\tau'\bar{\rho}=\rho+\omega y\ ,\qquad \tau'\bar{\sigma}=\sigma+\omega z\qquad\qquad (\omega=\tau''/\tau')\ ,$$

where $\bar{\rho}$, σ are respectively the transforms of ρ, σ. Let us consider any point $\rho+hy$ on the line $y\rho$, and any point $\sigma+kz$ on the line $z\sigma$, and let us impose the condition that the line joining these two points be a generator of the osculating regulus. Substituting the local coordinates h, λk, 1, λ of an arbitrary point on this line in equation (43) we find that the condition to be imposed is $h=k$. We now observe that if $\omega=h$, the transformed line $\bar{\rho}\bar{\sigma}$ coincides with this arbitrary generator of the osculating regulus. We thus have proved the theorem:

The line $\rho\sigma$ can be made to coincide with any prescribed generator of the osculating regulus by choosing the independent variable suitably.

Let the generator yz of the ruled surface R now vary over R. There is clearly a one-parameter family of osculating reguli, and the totality of their generators is a two-parameter family, or congruence, of lines called *the flecnode congruence* of the surface R (see Exs. 6, 7, 8). With a suitable definite choice of the independent variable, the locus of the line $\rho\sigma$ is an arbitrary ruled surface of this congruence, which is called *a derivative ruled surface* of the original ruled surface R, and is sometimes called the derivative ruled surface of R with respect to the independent variable employed. We remark incidentally that the totality of the asymptotic tangents of the surface R is also a congruence, called *the asymptotic congruence* of R.

13. The flecnode curves and the flecnode transformation. Continuing the geometrical investigation of the preceding section, we shall occasionally find it convenient to denote a ruled surface with director curves C_y, C_z by R_{yz}. Among the asymptotic tangents at the points of a generator l_{yz} of the surface R_{yz} there are two that are distinguished from the rest by the property of intersecting more consecutive generators than is usual for an asymptotic tangent. The points of contact of these two tangents are called *flecnode points*, or sometimes simply *flecnodes*, because at such a point the tangent plane of the surface R_{yz} intersects R_{yz} in a curve having at the point an inflexion and a node (see § 17, Chap. III). In the present section we shall study the locus of these flecnode points, called *the flecnode curves*, and on the basis of the flecnodal properties of a ruled surface shall define a transformation of ruled surfaces called *the flecnode transformation.*

The definition of a flecnode point may be concisely stated as follows. *A flecnode point on a ruled surface in ordinary space is a point at which the asymptotic tangent intersects four consecutive generators.* Since four skew lines in ordinary space have two straight line intersectors,* it follows that there are two flecnode points on each generator of a ruled surface in ordinary space. These two flecnode points may, of course, be real and distinct, real and coincident, or imaginary. Demanding that equation (39) be satisfied by the power series (40) for $x_1, \ldots, x_4$ conditionally in h, k and identically in Δt as far as the terms in Δt^3 we obtain, using the formulas (25),

$$u_{12}h^2-(u_{11}-u_{22})hk-u_{21}k^2=0 \ . \tag{44}$$

If h/k is a root of this equation, then the point $hy+kz$ is a flecnode point on the generator l_{yz}. Multiplying together the two linear expressions in y, z with

* Snyder and Sisam, 1914. 1, p. 143.

irrational coefficients thus obtained for the flecnodes, we find for the analytic representation of the flecnodes a quadratic expression with rational coefficients. This expression is a covariant, i.e., is invariant under the total transformation (21), (31), as is evident from its geometrical definition, and can be proved analytically. Thus we obtain the following theorem:

The flecnode points on a generator l_{yz} of a ruled surface R_{yz} in ordinary space are given by the factors of the covariant

$$u_{12}z^2+(u_{11}-u_{22})yz-u_{21}y^2 \; . \tag{45}$$

The flecnode points are coincident in case $\theta_4=0$. Thus we obtain the geometrical significance of this equation promised in Section 11. We shall exclude this case and suppose hereinafter, unless the contrary is stated, that the flecnodes are distinct, so that $\theta_4 \neq 0$.

The locus of the flecnode points on a ruled surface is a curve which ordinarily intersects each generator twice. Since in the neighborhood of such a generator there are two distinct parts of the locus, we shall speak of the locus as *the flecnode curves.* These are the director curves C_y, C_z in case $u_{12}=u_{21}=0$. It follows, moreover, from the definition of a flecnode point that every straight line intersecting ∞^1 generators of a ruled surface in space S_3 is a flecnode curve. Therefore *every point on a generator of a quadric surface is a flecnode point.* Consequently, if the surface R_{yz} is a quadric, the covariant (45) is indeterminate, so that for a quadric we have the necessary conditions $u_{12}=u_{21}=u_{11}-u_{22}=0$, which may be shown by the reader to be also sufficient by showing that when they are satisfied the curved asymptotics (37) are straight lines.

At a flecnode point *the tangent of the flecnode curve* is an ordinary two-point tangent whose locus is a developable surface with the flecnode curve as edge of regression, whereas the asymptotic tangent is a four-point tangent, called *the flecnode tangent,* whose locus is ordinarily a non-developable ruled surface called *a flecnode surface.* In order to find the parametric vector equations of the flecnode surfaces of a ruled surface R_{yz}, let us suppose that system (20) is in the canonical form for which

$$\left\{\begin{array}{lll} u_{12}=u_{21}=p_{11}=p_{22}=0 \; , & p'_{12}=2q_{12} \; , & p'_{21}=2q_{21} \; , \\ \theta_4=16(q_{11}-q_{22})^2\neq 0 \; , & \rho=2y'+p_{12}z \; , & \sigma=2z'+p_{21}y \; . \end{array}\right. \tag{46}$$

Now the director curves C_y, C_z are the flecnode curves, as in Figure 8. At a point P_y the tangent of the flecnode curve is determined by the point y',

and the flecnode tangent by the point ρ. The parametric vector equations of the flecnode surfaces are therefore, respectively,

$$\varphi=\rho+vy\ , \qquad \psi=\sigma+wz\ , \tag{47}$$

the points φ, ψ being the generating points, and v, w being scalar parameters.

If $p_{12}=0$, the flecnode curve C_y reduces to a straight line, since the first of equations (20) becomes $y''+q_{11}y=0$. In this line coincide the tangent

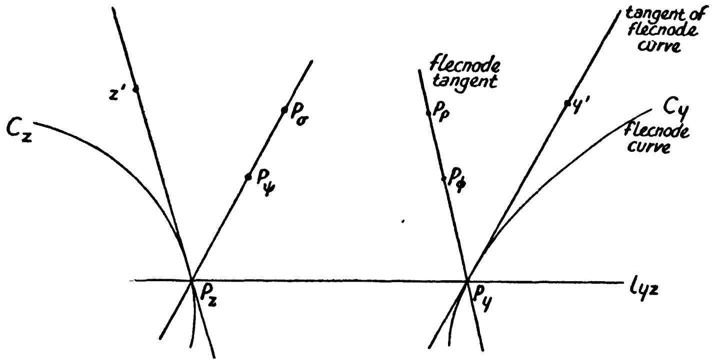

FIG. 8

yy' of the flecnode curve and the flecnode tangent $y\rho$, which in this case is fixed and does not generate a proper surface. We shall exclude this case and the similar case involving the line $z\sigma$, by supposing $p_{12}p_{21}\neq 0$.

The differential equations of the form (20) *for the flecnode surface* $R_{y\rho}$ generated by the line $y\rho$ are found by use of equations (20), (46) to be

$$\begin{cases} y''-2Q_{12}y'-\rho'-q_{11}y+Q_{12}\rho=0\ , \\ \rho''+P_{21}y'-2Q_{12}\rho'+Q_{21}y-q_{22}\rho=0\ , \end{cases} \tag{48}$$

where the coefficients denoted by capital letters are given by

$$Q_{12}=q_{12}/p_{12}\ , \qquad P_{21}=2(q_{11}+q_{22})-p_{12}p_{21}\ , \qquad Q_{21}=2q_{11}'-p_{12}q_{21}-4q_{11}Q_{12}\ .$$

For this system of equations the functions U_{ik}, analogous to the functions u_{ik} defined by the formulas (25), can be calculated. We find

$$\begin{cases} U_{12}=0\ , \qquad U_{21}=4(q_{22}'-q_{11}')+8q_{12}(q_{11}-q_{22})/p_{12}\ , \\ U_{11}-U_{22}=4(q_{11}-q_{22})\ . \end{cases} \tag{49}$$

The flecnodes on the line $y\rho$ are found, by writing formula (45) with U_{ik} in place of u_{ik} and with ρ in place of z, to be P_y and the point $(U_{11}-U_{22})\rho-U_{21}y$. Consequently *$C_y$ is a flecnode curve on the ruled surface $R_{y\rho}$ as well as on R_{yz}*. The line yz can be shown to be the flecnode tangent of the surface $R_{y\rho}$ at the point P_y by calculating the function P for system (48) analogous to the function ρ defined by the first of equations (41) for system (20). The result is $\mathrm{P}=-2Q_{12}y-p_{12}z$. By means of the substitution (27) it is easy to write the equations for the flecnode surface $R_{z\sigma}$, and to prove that C_z is a flecnode curve on $R_{z\sigma}$ as well as on R_{yz}.

We are now in position to define the flecnode transformation of ruled surfaces in ordinary space. The original surface R_{yz} has for its flecnode surfaces $R_{y\rho}$ and $R_{z\sigma}$. Moreover, $R_{y\rho}$ has for its flecnode surfaces R_{yz} and another, which has for its flecnode surfaces $R_{y\rho}$ and another, and so on. Thus we obtain from the ruled surface R_{yz} a sequence of ruled surfaces each of which has the two adjacent surfaces as flecnode surfaces. Such a sequence is called *a flecnode sequence*, and two surfaces in it are said to correspond by *the flecnode transformation.*

14. The osculating linear complex of a ruled surface in S_3. We first state the following definition. *The osculating linear complex along a generator l_{yz} of a ruled surface R_{yz} in ordinary space S_3 is the limit of the linear complex determined by l_{yz} and four neighboring generators of R_{yz} as each of these independently approaches l_{yz}, remaining on R_{yz}.* The local equation of the osculating linear complex will now be obtained in the line coordinates defined in Section 7; and a few consequences will be deduced.

Let us suppose that equations (20) are in the canonical form for which equations (46) are valid. Then differentiation of equations (20) and substitution from equations (41) enable us to express derivatives of y and z of any required order as linear combinations of y, z, ρ, σ. Actual calculation gives

$$y''=(p_{12}p_{21}/2-q_{11})y-q_{12}z-p_{12}\sigma/2\ ,$$

$$y'''=Ly+Mz+(p_{12}p_{21}-q_{11})\rho/2-3q_{12}\sigma/2\ ,$$

$$y''''=Ay+Bz+C\rho+D\sigma\ ,$$

where the coefficients L, A, B, C, will not be needed, and M, D are defined by

$$M=-q_{12}'+p_{12}(q_{11}+2q_{22}-p_{12}p_{21})/2\ ,$$

$$D=-2q_{12}'+p_{12}(q_{11}+q_{22}-p_{12}p_{21})/2\ .$$

As in Section 12 let us write Taylor's series for the coordinates of a point Y near the point P_y on the director curve C_y. Then substitution of the expressions just found for the derivatives of y enables us to write Y in the form

$$Y = y_1 y + y_2 z + y_3 \rho + y_4 \sigma ,$$

where the local coordinates $y_1, \ldots, y_4$ of the point Y are given by the expansions

$$\left\{\begin{aligned} y_1 &= 1 + (p_{12}p_{21} - 2q_{11})\Delta t^2/4 + \cdots , \\ y_2 &= -p_{12}\Delta t/2 - q_{12}\Delta t^2/2 + M\Delta t^3/6 + \cdots , \\ y_3 &= \Delta t/2 + (p_{12}p_{21} - q_{11})\Delta t^3/12 + \cdots , \\ y_4 &= -p_{12}\Delta t^2/4 - q_{12}\Delta t^3/4 + D\Delta t^4/24 + \cdots . \end{aligned}\right. \tag{50}$$

Similar expansions for the coordinates $z_1, \ldots, z_4$ of a point Z near P_z on C_z, referred to the local tetrahedron y, z, ρ, σ, can be written easily by means of the substitution (27) augmented by interchanging the subscripts 3 and 4. Then for the coordinates ω_{ik} of the line YZ, defined by

$$\omega_{ik} = y_i z_k - y_k z_i \qquad (i, k = 1, \ldots, 4) ,$$

we find the expansions

$$\left\{\begin{aligned} &\omega_{12} = 1 + \cdots , \qquad \omega_{13} = p_{21}(q_{11} - q_{22})\Delta t^4/48 + \cdots , \\ &\omega_{14} = \Delta t/2 + (p_{12}p_{21} - 3q_{11} - q_{22})\Delta t^3/12 + \cdots , \\ &\omega_{23} = -\Delta t/2 + (q_{11} + 3q_{22} - p_{12}p_{21})\Delta t^3/12 + \cdots , \\ &\omega_{42} = p_{12}(q_{22} - q_{11})\Delta t^4/48 + \cdots , \qquad \omega_{34} = \Delta t^2/4 + \cdots . \end{aligned}\right. \tag{51}$$

Writing the general equation of a linear complex and demanding that it be satisfied by the power series (51) for ω_{ik} identically in Δt as far as the terms in Δt^4, we find *the equation of the osculating linear complex,*

$$p_{12}\omega_{13} + p_{21}\omega_{42} = 0 . \tag{52}$$

The introduction of the osculating linear complex paves the way for further extensions of the theory of ruled surfaces. The null system of the osculating linear complex (52) has the equations

$$\xi_1 = p_{12}x_3 , \qquad \xi_2 = -p_{21}x_4 , \qquad \xi_3 = -p_{12}x_1 , \qquad \xi_4 = p_{21}x_2 . \tag{53}$$

At any point $(h, k, 0, 0)$ on the generator l_{yz} the equations of the tangent plane and the null plane are respectively

$$kx_3 - hx_4 = 0\,, \qquad p_{12}hx_3 - p_{21}kx_4 = 0\,.$$

These planes coincide in case $p_{12}h^2 - p_{21}k^2 = 0$. The two points $p_{12}z^2 - p_{21}y^2$ thus determined are called *the complex points* of the generator l_{yz}. At each of them the tangent plane of the surface corresponds to its point of contact in the null system of the osculating linear complex. The locus of the complex points is known as *the complex curves* on the surface. The two points $p_{12}z^2 + p_{21}y^2$ separate harmonically the complex points and also separate harmonically the flecnodes y, z. On this account they are called *the involute points* of the generator l_{yz}, and their locus is defined to be *the involute curves* on the surface.

The reader who wishes to refer to a fuller treatment of ruled surfaces in ordinary space than that which has been found here should consult* Wilczynski's book. Attention is especially directed to Chapter VIII on the conditions that a ruled surface may belong to a linear complex, or to a linear congruence, etc. Moreover, the reader who wishes to see a somewhat different method of handling the subject of ruled surfaces in ordinary space, namely, the method of differential forms, should consult† Chapter IV of the first volume of the treatise by Fubini and Čech. There are to be found in that chapter further geometrical results of great interest.

15. Quasi-asymptotic curves. An unspecialized ruled surface in a linear space of n dimensions $S_n (n>3)$ does not have on it an asymptotic curve other than the generators. The important rôle of the asymptotic curves on a ruled surface in space S_3 suggests the desirability of finding a substitute for them in the theory of ruled surfaces in hyperspace. Bompiani has defined on a ruled surface in space S_n certain curves, called *quasi-asymptotic curves*, with properties analogous to those of asymptotic curves. These quasi-asymptotic curves are the asymptotic curves when $n=3$, and so are actually a generalization of asymptotic curves. They will be discussed in this section.

At a point of a curve on a surface in space S_n the osculating plane of the curve and the tangent plane of the surface intersect in the tangent line of the curve, and therefore the ambient of these two planes is ordinarily a space S_3; but for an asymptotic curve the ambient is a plane, because the two planes coincide. This property of asymptotic curves suggests the following considerations which lead to the definition‡ of quasi-asymptotic

* Wilczynski, 1906. 1.

† Fubini and Čech, 1926. 1.

‡ Bompiani, 1924. 2, p. 313.

curves. At a point x on a ruled surface R_{yz} with parametric vector equation (14) in space S_n let us consider the space $S(k, 0)$, which is ordinarily a space S_{2k} determined by the $2k+1$ points (18); and let us also consider the osculating space S_{n-k} of a curve C_x, which is determined by the $n-k+1$ points $x, x', \ldots, x^{(n-k)}$. Wishing to discover the dimensionality of the ambient of these two spaces, we observe that the first $k+1$ of the latter set of points are linearly dependent on the points (18). Therefore the ambient of the space $S(k, 0)$ and the osculating space S_{n-k} is ordinarily the space S_n. *A quasi-asymptotic curve $A_{k,n-k}$ on a ruled surface is defined to be a curve such that at each of its points the ambient of the space $S(k, 0)$ of the surface and the osculating space S_{n-k} of the curve is a hyperplane S_{n-1}.* It is clear that this definition is valid only when $k>0$; moreover, when the space $S(k, 0)$ is a space S_{2k} one must also have $k<n/2$.

If the curve C_x is a quasi-asymptotic curve then the $n+1$ points in the set composed of the points (18) and $x^{(k+1)}, \ldots, x^{(n-k)}$ lie in a hyperplane and are therefore linearly dependent. Consequently the determinant of order $n+1$ made with their coordinates must vanish. Expressing this fact by writing only a typical row of the determinant within parentheses, we obtain *the differential equation of the quasi-asymptotics $A_{k,n-k}$* in the form

$$(54)\quad \begin{cases} (y, y', \ldots, y^{(k-1)}, z, z', \ldots, z^{(k-1)}, z^{(k)}+uy^{(k)}, z^{(k+1)}+uy^{(k+1)} \\ \quad +(k+1)u'y^{(k)}, z^{(k+2)}+uy^{(k+2)}+(k+2)u'y^{(k+1)} \\ \quad +(k+2)(k+1)u''y^{(k)}/2, \ldots, z^{(n-k)}+uy^{(n-k)}+(n-k)u'y^{(n-k-1)} \\ \quad +\cdots+C_{n-k,k}u^{(n-2k)}y^{(k)})=0\,. \end{cases}$$

Since this differential equation for u as a function of t is of order $n-2k$, it follows that *there are ∞^{n-2k} quasi-asymptotics $A_{k,n-k}$ on a ruled surface in space S_n.* One of these can be determined uniquely by assigning for it $n-2k$ points suitably chosen on the surface; or else by assigning an element E_{n-2k-1} at one point on the surface; or else in still other ways.

Placing $n=3$ and $k=1$, we observe that the curves $A_{1,2}$ are the curved asymptotics that exist on a ruled surface in ordinary space. More generally, if n is odd, let $n=2m-1$. If we now place $k=m-1$, then $n-k=m$ and there are observed to be *∞^1 curves $A_{m-1,m}$ on a ruled surface in space S_{2m-1}*; their differential equation is

$$(55)\qquad \begin{cases} (y, y', \ldots, y^{(m-2)}, z, z', \ldots, z^{(m-2)}, \\ z^{(m-1)}+uy^{(m-1)}, z^{(m)}+uy^{(m)}+u'my^{(m-1)})=0\,. \end{cases}$$

This equation for u as a function of t is of the first order; therefore one curve $A_{m-1,m}$ passes through each point of the surface. Moreover, this

equation is an equation of Riccati, so that we have an extension of the theorem of Serret:

The cross ratio of the four points in which four quasi-asymptotics $A_{m-1,m}$ on a ruled surface in space S_{2m-1} intersect a generator is the same for all generators of the surface.

If n is even, let $n=2m$. If we now place $k=m$, then $n-k=m$ and there is observed to be *a unique curve $A_{m,m}$ on a ruled surface in space S_{2m};* this curve is the locus of the point x given by equation (14) when u satisfies the equation

$$(y, y', \ldots, y^{(m-1)}, z, z', \ldots, z^{(m-1)}, z^{(m)}+uy^{(m)})=0 . \tag{56}$$

At points of this curve the space $S(k, 0)$ of the surface, which is determined by the points (18), is a space S_{2k-1} instead of the usual space S_{2k}, since the points (18) are precisely the points which equation (56) asserts to be linearly dependent. In particular, if $n=2$ and $k=1$, then $m=1$ and the unique curve $A_{1,1}$ on ∞^1 lines in a plane is their envelope. Finally, if $n=4$, $k=2$, then $m=2$ and the unique curve $A_{2,2}$ on a ruled surface in a space S_4 is the locus of the point (14) when u satisfies the equation

$$(y, y', z, z', z''+uy'')=0 . \tag{57}$$

EXERCISES

1. In ordinary space a developable and a curve are dual configurations. Compare Sections 6 and 9 from this point of view.

2. The equation of the developable of which the twisted cubic (I, 26) is the edge of regression is

$$4(x_1x_3-x_2^2)(x_2x_4-x_3^2)-(x_2x_3-x_1x_4)^2=0 .$$

This developable is of class 3 and order 4, the *class* of a developable being the number of its tangent planes through any point, and the *order* being the number of points in which a straight line meets it.

3. The cubic ruled surface $x_1^2x_3-x_2^2x_4=0$ has two distinct rectilinear directrices $y_1=0$, $y_2=0$, $y_3=t^2$, $y_4=1$ and $z_1=1$, $z_2=t$, $z_3=0$, $z_4=0$. The differential equations of the form (20) for this surface are $y''-y'/t=0$, $z''=0$. The asymptotic curves on the surface are unicursal quartics each of which crosses each generator in two points separating harmonically the flecnodes on the generator.

WILCZYNSKI, 1906. 1, p. 145

4. The cubic ruled surface $x_1(x_1x_3+x_2x_4)+x_2^3=0$ has one rectilinear directrix $y_1=0$, $y_2=0$, $y_3=-t$, $y_4=1$ and a director curve $z_1=-1$, $z_2=-t$, $z_3=0$, $z_4=t^2$. The

differential equations of the form (20) for this surface are $y''=0$, $z''+2ty'-2y=0$. The asymptotic curves are twisted cubics, while the flecnode curves coincide in the rectilinear directrix. WILCZYNSKI, 1906. 1, p. 145

5. Considering an integral surface of system (20) with $p_{ik}=0$, show that the local coordinates of a point near P_x on the curve C_x defined by equation (14) with $u=u(t)$, referred to the tetrahedron y, z, y', z', are given by

$$\begin{aligned}
x_1 &= u+u'\Delta t+(u''-q_{11}u-q_{21})\Delta t^2/2+\cdots,\\
x_2 &= 1-(q_{12}u+q_{22})\Delta t^2/2+\cdots,\\
x_3 &= u\Delta t+u'\Delta t^2+(3u''-q_{11}u-q_{21})\Delta t^3/6+\cdots,\\
x_4 &= \Delta t-(q_{12}u+q_{22})\Delta t^3/6+\cdots.
\end{aligned}$$

The equation of the tangent plane of the surface at P_x is $x_3-ux_4=0$; the equations of the tangent line of C_x are $x_3-ux_4=x_1-ux_2-u'x_4=0$; and the equation of the osculating plane of C_x is

$$2u'(x_1-ux_2-u'x_4)-(x_3-ux_4)[u''+q_{12}u^2-(q_{11}-q_{22})u-q_{21}]=0\ .$$

Every asymptotic curve $u=$const. has three coincident points in the tangent plane, and two in its tangent line. At a flecnode the asymptotic curve has four coincident points in the tangent plane and three in its tangent line, its osculating plane being indeterminate. A flecnode is therefore an inflexion point on the asymptotic curve through it.

6. By means of equations (20), (39) with $p_{ik}=0$ show that the points η, ζ defined by $\eta=y'+hy$, $\zeta=z'+hz$ determine a general generator $l_{\eta\zeta}$ of the osculating regulus along a generator l_{yz} of the ruled surface R_{yz}, so that the point φ defined by $\varphi=\eta+\lambda\zeta$ is a general point on $l_{\eta\zeta}$. If, as t varies, $l_{\eta\zeta}$ describes a developable of the flecnode congruence of R_{yz}, and if P_φ is the corresponding focal point of $l_{\eta\zeta}$, then φ' must be a linear combination of y and z only; hence we obtain two conditions:

$$h'-h^2-q_{11}-q_{21}\lambda=0\ ,\qquad \lambda(h'-h^2)-q_{12}-q_{22}\lambda=0\ .$$

Eliminating $h'-h^2$, prove that $l_{\eta\zeta}$ has two such foci which are on the flecnode tangents, so that the lines of the flecnode congruence are tangent to both of the flecnode surfaces, the points of contact of each line being its foci. (On this account the flecnode surfaces are called *the focal surfaces* of the flecnode congruence. Moreover, *the flecnode congruence* receives its name from its relation to the flecnode surfaces.) Eliminating λ, prove that the lines of the flecnode congruence can be assembled into ∞^1 developables in two ways.

7. By means of equations (37), (47) show that the differential equation of the asymptotic curves on the integral ruled surface $R_{y\rho}$ of the equations (48) is $2v'=2(q_{11}+q_{22})-p_{12}p_{21}+v^2$; show that the equation of the asymptotics on the surface

$R_{z\sigma}$ is the same with v replaced by w. The asymptotics on $R_{y\rho}$ and $R_{z\sigma}$ therefore correspond in the sense that, if a point of contact of a generator of the flecnode congruence describes an asymptotic curve on one surface, the other point of contact does also on the other. (A congruence having this property is called *a W congruence.*)

8. Using the canonical form of system (20) described by equations (46), and placing $A=(q'_{11}-q'_{22})/(q_{11}-q_{22})$, $B=q_{12}/p_{12}$, show that the point $\rho+Ay/2$ on the flecnode tangent $l_{y\rho}$ at the point P_y is the harmonic conjugate of P_y with respect to the other flecnode $\rho+(A-2B)y$ on $l_{y\rho}$ and the point $\rho+2By$ which lies on the same generator of the osculating regulus $x_2x_3-x_1x_4=0$ as the point $\sigma+2Bz$ where the flecnode tangent $l_{z\sigma}$ at the point P_z meets the osculating plane $p_{12}x_2+p_{12}^2x_3-2q_{12}x_4=0$ of the flecnode curve C_y at P_y. The point $\sigma+Az/2$ can be defined similarly on the line $l_{z\sigma}$. The line joining these two points is a generator of the osculating regulus and generates a ruled surface of the flecnode congruence (called *the principal ruled surface of the flecnode congruence*); this is the derivative ruled surface $R_{\rho\sigma}$ in case the independent variable is chosen so that $\theta_4=\text{const.}$

9. In a linear space of four dimensions S_4 three skew lines have just one straight line intersector.

10. In space S_4 the ruled surface defined, except for a projective transformation, by the differential equations

$$y'''=\alpha y+\beta z+\gamma y'+\delta z'+\epsilon y'' \ ,$$
$$z''=ay+bz+cy'+dz'+ey'' \ ,$$

has on it an asymptotic curve if, and only if, $2e'=c+ed$. The unique quasi-asymptotic $A_{2,2}$ on the surface is the locus of the point z–ey at which a line can be drawn intersecting three consecutive generators. The osculating plane at a point of the curve $A_{2,2}$ lies in the tangent space S_3 of the surface along the generator through the point. This space S_3 is the hyperplane employed in the definition of the curve $A_{2,2}$. If there is an asymptotic curve on the surface this curve is the curve $A_{2,2}$.

11. From the equations of Exercise 10 obtain the equation

$$z'''=Ay+Bz+Cy'+Dz'+Ey'' \ ,$$

where

$$A=a'+ad+e\alpha \ , \qquad B=b'+bd+e\beta \ , \qquad C=c'+a+cd+e\gamma \ ,$$
$$D=d'+b+d^2+e\delta \ , \qquad E=e'+c+ed+e\epsilon \ .$$

Then show that the differential equation of the ∞^2 quasi-asymptotics $A_{1,3}$ on the ruled surface considered in that exercise can ordinarily be written in the form

$$(u+e)[C+u\gamma+3u''-u(D+u\delta)]-(c+2u'-ud)(E+u\epsilon+3u')=0 \ .$$

The osculating space S_3 at a point of one of these curves contains the tangent plane of the surface at the point. A curve C_x on the surface is a quasi-asymptotic $A_{1,3}$ in case at each of its points the space $S(2, 1)$ of the surface in the direction of the curve has four-point contact with the curve.

12. In space S_4 the tangent space S_3 along a variable generator of a ruled surface osculates a curve (called *the associated curve* of the surface). If the quasi-asymptotic $A_{2,2}$ is chosen for the curve C_z, so that $e=0$, the associated curve is generated by the point X defined by

$$X=c(cy-z')+(a+cd-c')z\ ,$$

in the notation of Exercise 10. If the curve $A_{2,2}$ is not an asymptotic curve, the tangent line at a point of the associated curve intersects the corresponding generator yz at the point z, where the generator crosses $A_{2,2}$; and the osculating plane of the associated curve contains the generator. If the curve $A_{2,2}$ is an asymptotic curve but not a straight line, then the associated curve coincides with $A_{2,2}$.

BOMPIANI, 1914. 2, p. 309

13. At a point of a flecnode curve of a ruled surface R in space S_3, consider the harmonic conjugate line l of the tangent of the flecnode curve with respect to the generator of the surface and the flecnode tangent. When the generator varies over the surface R, the line l generates a developable surface. Determine the edges of regression of the two developables thus associated with the surface R.

WILCZYNSKI, 1906. 1, p. 233

14. There exists a one-parameter family of ruled surfaces having one flecnode curve in common with a ruled surface in space S_3. The surfaces of this family can be paired as in an involution, the double surfaces being the developable of tangents of the flecnode curve and the developable associated with this curve as described in Exercise 13. At each point of the flecnode curve the generators of the ruled surfaces are in involution. WILCZYNSKI, 1906. 1, p. 233

15. A ruled surface in space S_5, which has on it two curves C_y, C_z such that their osculating planes at points on each generator do not intersect, is an integral surface of a pair of equations of the form

$$y'''+3p_{11}y''+3p_{12}z''+3q_{11}y'+3q_{12}z'+r_{11}y+r_{12}z=0\ ,$$
$$z'''+3p_{21}y''+3p_{22}z''+3q_{21}y'+3q_{22}z'+r_{21}y+r_{22}z=0\ ,$$

Study the surface by means of these equations. STOUFFER, 1913. 3, p. 204

16. The two osculating planes of the flecnode curves, the two osculating planes of the complex curves, and the two osculating planes of the involute curves, at points of a generator of a ruled surface in space S_3, belong to a pencil if, and only if, in the notation of equations (46),

$$p_{21}^2(p_{12}^2q_{22}-p_{12}q_{12}'+3q_{12}^2)=p_{12}^2(p_{21}^2q_{11}-p_{21}q_{21}'+3q_{21}^2)\ .$$

CARPENTER, 1928. 7, p. 481

17. The line of intersection of the two osculating planes of the flecnode curves at the points where these curves cross a generator l of a ruled surface R in space S_3, the line that corresponds to this line in the null system of the osculating linear complex of R along l, and the generator l itself determine a quadric surface. This quadric intersects the osculating quadric of R along l in l and in a twisted cubic which passes through the complex points of l. The point corresponding to a general osculating plane of this cubic in the null system of the osculating linear complex of R along l generates another twisted cubic whose points are in one-to-one correspondence with the points of the first cubic. The lines joining corresponding points of these cubics form a ruled surface T, upon which the two cubics are asymptotic curves, and which belongs to a linear congruence, whose directrices cut each generator of the surface T in two points separating harmonically the points in which the two cubics cross the generator. CARPENTER, 1923. 5, pp. 108 and 111

18. As a generator l of a ruled surface R, not a quadric, in space S_3 varies over the surface R, the envelope of the osculating quadric of R along the generator l consists of the surface R itself and of the two flecnode surfaces of R.

19. Starting with the canonical form of system (20) for which $p_{ik}=0$, calculate the linear differential equation of the sixth order satisfied by the line coordinates of a generator of an integral ruled surface. Hence deduce conditions necessary and sufficient that an integral ruled surface of system (20) belong to a single non-special linear complex, to a single special linear complex, to a linear congruence with distinct directrices, or to a linear congruence with coincident directrices.

WILCZYNSKI, 1906. 1, Chap. VIII

20. The most general transformation (21) that preserves the canonical form of system (20) characterized by the conditions $p_{ik}=0$ $(i, k=1, 2)$ has α, β, γ, δ all constants.

21. Consider the hyperplane S_{n-1} associated with each point of a quasi-asymptotic curve $A_{k,n-k}$ on a ruled surface in space S_n. All these hyperplanes associated with the points of a curve $A_{k,n-k}$ are the osculating hyperplanes of a curve $B_{k,n-k}$ associated with $A_{k,n-k}$. Discuss the relations of these two curves, showing that each osculating space S_{n-1-i} of the curve $B_{k,n-k}$ contains the corresponding osculating space S_{n-k-i} of the curve $A_{k,n-k}$ $(i=0, \ldots, n-\text{k})$.

BOMPIANI, 1914. 2, p. 315

22. If a developable surface in space S_n has on it a plane curve (not the edge of regression and not a generator), the developable is immersed in ordinary space. In general, if a developable surface in space S_n has on it a curve (not the edge of regression and not a generator) immersed in a space S_k, then the developable is immersed in a space S_{k+1} $(0<k<n)$.

23. (A ruled surface in space S_n is said to have *indices of developability h, k* when the generators of the surface lie one by one in the osculating spaces S_h of a curve and each generator intersects the osculating space S_k which is in the osculating space S_h containing it.) For an ordinary ruled surface in space S_3 the indices are 2, 1. For a developable surface the indices are 2, 0. For an ordinary ruled surface in space S_4 the indices (see Ex. 12) are 2, 1; and if a ruled surface in space S_4 has on it an asymptotic curve not a straight line, the indices are 2, 0. The maximum values of the indices are $n/2$, $(n-1)/2$ if n is even, and are $(n+1)/2$, $n-2$ if n is odd.

BOMPIANI, 1914. 2, p. 312

CHAPTER III

SURFACES IN ORDINARY SPACE

Introduction. There are two well-recognized methods that have been used in developing extensive analytic theories of projective differential geometry. Since these methods are perhaps best exemplified by the theories of analytic non-ruled surfaces in ordinary space, it seems appropriate to make a few comments on them here. These methods may be designated as *the American method,* which is due to Wilczynski, and *the Italian method,* which is due to Fubini.

The method of Wilczynski originated, as has already been indicated, in connection with curves and ruled surfaces in the years immediately preceding 1906, the earliest memoir of interest in this connection appearing in 1901. The method was extended about 1907 so as to be available for the study of so-called *curved surfaces* in ordinary space. In its most general form this method can be described as follows. When a configuration is to be studied, one starts with the parametric equations of the configuration, which express the projective homogeneous coordinates of a variable element of the configuration as functions of a certain number of parameters. The first step is to establish a completely integrable system of linear homogeneous partial differential equations of which these coordinates constitute a fundamental set of solutions, the independent variables in the equations being the said parameters. These equations define the configuration except for a projective transformation, since the most general solution of the equations can be expressed as a linear combination of the fundamental set of solutions with constant coefficients. The next step is to determine the most general group of transformations of the dependent and independent variables in the equations that does not disturb the configuration, and to calculate the effect of this group of transformations on the differential equations. The third step is to find a complete and independent system of invariants and covariants; invariants are functions of the coefficients of the differential equations and of their derivatives, and covariants are functions of these and also of the dependent variables and their derivatives, that are unchanged by the group of transformations, except possibly for a factor depending only on the transformations. This work is purely analytic and may employ the Lie theory of continuous groups. Finally, the invariants and covariants are interpreted geometrically and used to study the geometry of the configuration in the manner explained in Section 2 in connection with curves.

The method of Fubini may be said to have originated just before 1916. In fact, the method seems to have been introduced in a memoir published in 1914, in which Fubini proposed the problem of defining a surface except for a projective transformation by means of differential forms. The analytic basis for the study of any configuration by this method is a system of differential forms, which define the configuration except for a projective transformation. The absolute calculus of Ricci may be employed in dealing with these forms. But when it is a question of finding the coordinates of a variable element of the configuration defined by the forms, the differential equations that Wilczynski would use as fundamental appear.

The reader who is familiar with the metric differential geometry of surfaces in ordinary space will see that the two methods now under consideration could be used in developing this theory. Wilczynski would start with the differential equations of Gauss for the coordinates of a variable point on a surface and the direction cosines of the normal to the surface at the point. The condition of Gauss on the fundamental coefficients and the two equations of Codazzi would enter the theory as conditions of complete integrability of the differential equations, and the first and second quadratic differential forms would appear later. Fubini would start, as Gauss did, with the quadratic differential forms as fundamental and would afterward obtain the differential equations, the conditions of Gauss and Codazzi, and the fundamental theorem on the determination of a surface when the six fundamental coefficients are given.

Each method has its own advantages. Their starting points are different but the two theories soon become interlaced. Throughout most of this book a modified form of the American method will be used, the computation of complete systems of invariants and covariants being omitted. The reader who is interested in this phase of the theory may consult the original memoirs cited hereinafter, and anyone interested in the Italian method may refer to Section 59 in Chapter VIII and to the treatise* by Fubini and Čech.

The theory of non-developable surfaces in ordinary space is much simplified by taking the asymptotic curves as parametric. Moreover, the fundamental system of differential equations is susceptible of various canonical forms, each of which has its own advantages. The canonical form used by Wilczynski is analytically simple but lacks certain invariantive properties possessed by the canonical form of Fubini. The latter form will be used throughout this chapter unless the contrary is expressly stated, in spite of the fact that ruled surfaces are thereby excluded from consideration. The

* Fubini and Čech, 1926. 1 and 1927. 1.

connection between Wilczynski's and Fubini's canonical forms is discussed further in Exercise 2 and in Section 60.

The contents of the present chapter may be outlined as follows. After establishing an analytic basis for the theory of surfaces referred to their asymptotic curves in ordinary space, power series expansions are calculated for the local coordinates of a point on the surface, and some immediate geometrical applications thereof are made. Following the first two sections thus occupied come eight sections each of which is devoted to some phase of the general subject. Some of the many quadrics that have been associated with a point of a surface are discussed. Then the general theory of pairs of rectilinear congruences in a certain *reciprocal* relation with respect to a surface unifies the discussion of several particular covariant pairs of such congruences; this leads to the definition of a pair of flat pencils, called *canonical pencils*, covariantly associated with each point of a surface. Some important types of families of curves on a surface are considered under the headings of *conjugate nets* and of *hypergeodesics*. *Čech's transformation*, which comes next, is a point-plane correspondence associated with a point of a surface. Some more families of curves on a surface are considered under the general title of *pangeodesics*, and this chapter closes with a discussion of a certain envelope associated with a surface; this envelope is the locus of the vertices of a tetrahedron called *the tetrahedron of Demoulin*.

16. The differential equations of a surface referred to its asymptotic net. The first problem considered in this section is to establish *the defining system of differential equations* of a surface referred to its asymptotic net. Then one of the *integrability conditions* of these equations enables us to reduce the equations to *Fubini's canonical* form by means of a transformation of the dependent variable. The integrability conditions for this form can be written very simply by means of certain useful formulas. This section closes with the determination of the residual group of transformations leaving Fubini's canonical form invariant, and the calculation of the effect of this group of transformations on the coefficients of the differential equations.

The defining system of differential equations that are fundamental in this chapter may be established in the following way. Let us consider an analytic non-developable proper surface S with the parametric vector equation $x=x(u, v)$ immersed in ordinary space S_3; and let us suppose that the asymptotic net on S is parametric. Equations (II, 9) show that under this assumption we have

$$L=N=0\,, \qquad M\neq 0\,,$$

where L, M, N are defined in equations (II, 10). Since the columns of a vanishing determinant are linearly dependent, it follows that *the four coordinates x of a general point on the surface S are solutions of a system of equations* of the form*

$$\text{(1)} \qquad \begin{cases} x_{uu} = px + \alpha x_u + \beta x_v \ , \\ x_{vv} = qx + \gamma x_u + \delta x_v \ , \end{cases}$$

wherein the coefficients are scalar functions of u, v, and subscripts indicate partial differentiation. Moreover, *the coordinates x are not solutions of any equation of the form*

$$\text{(2)} \qquad Ax_{uv} + Bx_u + Cx_v + Dx = 0$$

whose coefficients are scalar functions of u, v not all zero. We may remark that the asymptotic character of the parametric curves is rendered apparent by the form of equations (1) if the definition of an asymptotic curve is kept in mind. Moreover, singular points on the surface S at which an equation of the form (2) may happen to be conditionally satisfied will be avoided.

The coefficients of system (1) are not arbitrary, but must satisfy certain partial differential equations, called integrability conditions, which we proceed to discuss. The four third derivatives of x can be calculated from system (1), and can be expressed uniquely as linear combinations of x, x_u, x_v, x_{uv} by differentiation and substitution. But of the five fourth derivatives of x there are three that can be calculated in two ways, namely, x_{uuuv}, x_{uuvv}, x_{uvvv}; however, since the order of differentiation is immaterial, we must have

$$(x_{uu})_{uv} = (x_{uu})_{vu} \ , \qquad (x_{uu})_{vv} = (x_{vv})_{uu} \ , \qquad (x_{vv})_{uv} = (x_{vv})_{vu} \ .$$

The first and third of these equations are formal identities. But after some calculation the second equation reduces to an equation of the form (2), and then all four of its coefficients must be zero. Thus we obtain *four integrability conditions* on the coefficients of system (1). Of these conditions the only one that is needed immediately is the one that results from the vanishing of the coefficient of x_{uv}. Neglecting for the moment the other three conditions and calculating only this one, we find it to be

$$\alpha_v = \delta_u \ .$$

* Wilczynski, 1907. 1, p. 244.

It follows from this condition that *there exists a function θ defined, except for an arbitrary additive constant, by the differential equations*

$$\theta_u = a \ , \qquad \theta_v = \delta \ .$$

When the four integrability conditions are satisfied, it turns out that every derivative of x can be expressed uniquely as a linear combination of x, x_u, x_v, x_{uv}.

It is clear that the surface S with which we started is not the only integral surface of system (1); in fact, any surface into which S can be projected is equally an integral surface of (1). Conversely, *system* (1) *with the four integrability conditions satisfied defines a surface in ordinary space except for a projective transformation,* since the theory of differential equations tells us that system (1) in the presence of the integrability conditions has four linearly independent solutions forming a fundamental set, and that the most general solution of (1) is a linear combination of these four solutions with constant coefficients. Such a system of partial differential equations is said to be *completely integrable.*

We shall next show how, by a suitable choice of proportionality factor for the homogeneous coordinates, to obtain *Fubini's canonical form* of system (1), and shall afterward write the integrability conditions explicitly therefor. The effect of the transformation

$$x = \lambda(u, v)\bar{x} \qquad (\lambda \text{ scalar}) \tag{3}$$

on system (1) is to produce another system of equations of the same form whose coefficients, indicated by dashes, are given by

$$\begin{cases} \lambda\bar{p} = -\lambda_{uu} + p\lambda + \theta_u\lambda_u + \beta\lambda_v \ , \\ \lambda\bar{q} = -\lambda_{vv} + q\lambda + \gamma\lambda_u + \theta_v\lambda_v \ , \\ \bar{\beta} = \beta \ , \qquad \bar{\gamma} = \gamma \ , \qquad \bar{\theta} = \theta - 2\log\lambda \ , \end{cases} \tag{4}$$

where $\bar{\theta}$ is determined except for an arbitrary additive constant. The form of equations (1) shows that *the u-curves* (*one family of asymptotics*) *on the surface S are straight lines if $\beta = 0$, and the v-curves* (*the other family of asymptotics*) *are straight lines if $\gamma = 0$.* If we assume that the surface S is not ruled, so that $\beta\gamma \neq 0$, and if we choose λ so that

$$\lambda^2\beta\gamma = e^{\theta} \ , \tag{5}$$

then $\bar{\theta}=\log \bar{\beta}\bar{\gamma}$. Thus we obtain *Fubini's canonical form of the differential equations of a non-ruled surface referred to its asymptotic curves*, namely,

$$\tag{6}\begin{cases} x_{uu}=px+\theta_u x_u+\beta x_v\ , \\ x_{vv}=qx+\gamma x_u+\theta_v x_v \end{cases} \qquad (\theta=\log \beta\gamma)\ .$$

We shall use this canonical system of equations as fundamental throughout this chapter unless otherwise indicated. The coordinates x, when multiplied by the factor λ defined by equation (5), are called* *Fubini's normal coordinates.*

The coefficients of system (6) satisfy three integrability conditions which we proceed to write. The formulas for the third derivatives of x expressed as linear combinations of x, x_u, x_v, x_{uv} are found from system (6) to be

$$\tag{7}\begin{cases} x_{uuu}=(p_u+p\theta_u)x+(p+\theta_u^2+\theta_{uu})x_u+(\beta_u+\beta\theta_u)x_v+\beta x_{uv}\ , \\ x_{uuv}=(p_v+\beta q)x+(\beta\gamma+\theta_{uv})x_u+\pi x_v+\theta_u x_{uv}\ , \\ x_{uvv}=(q_u+\gamma p)x+\chi x_u+(\beta\gamma+\theta_{uv})x_v+\theta_v x_{uv}\ , \\ x_{vvv}=(q_v+q\theta_v)x+(\gamma_v+\gamma\theta_v)x_u+(q+\theta_v^2+\theta_{vv})x_v+\gamma x_{uv}\ , \end{cases}$$

where π, χ are defined by the formulas

$$\tag{8}\pi=p+\beta\psi\ , \qquad \chi=q+\gamma\varphi\ ,$$

and φ, ψ by

$$\tag{9}\varphi=(\log \beta\gamma^2)_u\ , \qquad \psi=(\log \beta^2\gamma)_v\ .$$

The integrability conditions of equations (6) can now without difficulty be shown by the method explained above to be

$$\tag{10}\begin{cases} \theta_{uvv}=(\gamma\varphi)_u+2q_u+\theta_v\theta_{uv}-\beta\gamma\psi\ , \\ \theta_{uuv}=(\beta\psi)_v+2p_v+\theta_u\theta_{uv}-\beta\gamma\varphi\ , \\ p_{vv}-\theta_v p_v+\beta q_v+2q\beta_v=q_{uu}-\theta_u q_u+\gamma p_u+2p\gamma_u\ . \end{cases}$$

If two functions l, m are defined by placing

$$l=2p+\beta\psi+\theta_u^2/2-\theta_{uu}\ ,$$
$$m=2q+\gamma\varphi+\theta_v^2/2-\theta_{vv}\ ,$$

* Fubini, 1918. 1, p. 1033.

the integrability conditions (10) can be written in the form

$$l_v=\beta\gamma\varphi\ ,\qquad m_u=\beta\gamma\psi\ ,$$
$$\beta_{vvv}-\beta m_v-2m\beta_v=\gamma_{uuu}-\gamma l_u-2l\gamma_u\ .$$

The most general transformation (3) *which preserves Fubini's canonical form has* $\lambda=$ const., as can be verified on inspection of equation (5). *But any transformation of the independent variables that leaves the asymptotic net parametric also leaves Fubini's canonical form invariant.* In fact, aside from a mere interchange of the two asymptotic families, effected by a transformation $\bar{u}=v$, $\bar{v}=u$, the most general transformation of parameters leaving the asymptotic net invariant is of the form $\bar{u}=U(u)$, $\bar{v}=V(v)$. Let us consider the total transformation

$$x=c\bar{x}\ ,\qquad \bar{u}=U(u)\ ,\qquad \bar{v}=V(v)\qquad (c=\text{const.};\ cU'V'\neq 0)\ . \tag{11}$$

The effect of this transformation on the coefficients of equations (6) is found to be given by

$$\begin{cases}\bar{p}=p/U'^2\ , & \bar{\beta}=\beta V'/U'^2\ , \\ \bar{q}=q/V'^2\ , & \bar{\gamma}=\gamma U'/V'^2\ , \qquad \bar{\theta}=\theta-\log U'V'\ ,\end{cases} \tag{12}$$

where $\bar{\theta}$ is as usual determined except for an arbitrary additive constant. It is now easy to verify that $\bar{\theta}=\log\bar{\beta}\bar{\gamma}$ and thus to show that Fubini's canonical form is preserved. Moreover, it could be shown that *the transformation* (11) *is the most general transformation that leaves Fubini's canonical form invariant.*

17. The local coordinate system. Power series expansions. The study of the geometry of a surface in the neighborhood of one of its points P_x is facilitated by the use of a local coordinate system intimately connected with the surface at this point. Power series expansions can be calculated for the homogeneous local coordinates of a point X near the point P_x on the surface. Then one non-homogeneous local coordinate of the point X can be expanded as a power series in the other two coordinates. This series, obtained in the early part of this section, will be employed in the latter part for deriving some geometrical properties of the surface. In particular, the points called *flecnode points* will be defined, and their definition will be connected with the definition of flecnode points on ruled surfaces as employed in Chapter II.

Let us consider an integral surface S of equations (6). The points x, x_u, x_v, x_{uv} are easily shown by simple calculations to be covariant under the

transformation (11), and are ordinarily not coplanar. We shall use these points as the vertices of a *local tetrahedron of reference* at the point P_x, with a unit point chosen so that any point whose coordinates in the original system are given by an expression of the form

$$x_1x + x_2x_u + x_3x_v + x_4x_{uv}$$

shall have local coordinates proportional to $x_1, \ldots, x_4$. The effect of the transformation (11) on this local coordinate system is to change the unit point according to the formulas

$$\bar{x}_1 = x_1 \,, \qquad \bar{x}_2 = U'x_2 \,, \qquad \bar{x}_3 = V'x_3 \,, \qquad \bar{x}_4 = U'V'x_4 \,. \tag{13}$$

The unit point is therefore not ordinarily a covariant point. Two of the edges of the tetrahedron are the asymptotic tangents at the point P_x. The complete geometric description of the tetrahedron will appear in Section 20.

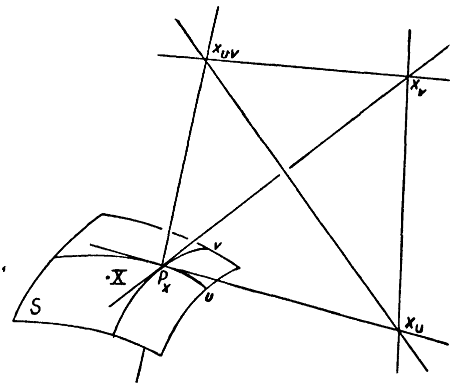

FIG. 9

The coordinates of any point X near a point P_x on a surface S as sketched in Figure 9 can be represented by Taylor's expansion as power series in the increments Δu, Δv corresponding to displacement on S from P_x to the point X:

$$X = x + x_u\Delta u + x_v\Delta v + (x_{uu}\Delta u^2 + 2x_{uv}\Delta u\Delta v + x_{vv}\Delta v^2)/2 + \cdots \,. \tag{14}$$

It is possible by means of equations (6), (7) and the equations obtained therefrom by differentiation, together with the integrability conditions

(10), to express every derivative of x uniquely as a linear combination of x, x_u, x_v, x_{uv}. Therefore X can be expressed uniquely in the form

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv} ,$$

in which $x_1, \ldots, x_4$ are power series in Δu, Δv which represent the local coordinates of the point X. Performing the calculations thus indicated leads to the following *power series expansions of the local coordinates of a point near the point P_x on the surface S referred to the tetrahedron x, x_u, x_v, x_{uv}, with suitably chosen unit point:*

$$(15)\quad \begin{cases} x_1 = 1 + (p\Delta u^2 + q\Delta v^2)/2 + \cdots , \\ x_2 = \Delta u + (\theta_u \Delta u^2 + \gamma \Delta v^2)/2 + [(p + \theta_u^2 + \theta_{uu})\Delta u^3 + 3(\beta\gamma + \theta_{uv})\Delta u^2 \Delta v \\ \qquad + 3\chi \Delta u \Delta v^2 + (\gamma_v + \gamma\theta_v)\Delta v^3]/6 + \cdots , \\ x_3 = \Delta v + (\beta \Delta u^2 + \theta_v \Delta v^2)/2 + [(\beta_u + \beta\theta_u)\Delta u^3 + 3\pi \Delta u^2 \Delta v \\ \qquad + 3(\beta\gamma + \theta_{uv})\Delta u \Delta v^2 + (q + \theta_v^2 + \theta_{vv})\Delta v^3]/6 + \cdots , \\ x_4 = \Delta u \Delta v + (\beta \Delta u^3 + 3\theta_u \Delta u^2 \Delta v + 3\theta_v \Delta u \Delta v^2 + \gamma \Delta v^3)/6 + [(\beta_u + \beta\theta_u)\Delta u^4 \\ \qquad + 2(\pi + \theta_u^2 + \theta_{uu})\Delta u^3 \Delta v + 3(2\theta_{uv} + \theta_u\theta_v + \beta\gamma)\Delta u^2 \Delta v^2 \\ \qquad + 2(\chi + \theta_v^2 + \theta_{vv})\Delta u \Delta v^3 + (\gamma_v + \gamma\theta_v)\Delta v^4]/12 + \cdots . \end{cases}$$

The series (15) are fundamental in much of what follows. It is understood that the point X is sufficiently near to the point P_x so that all the power series envisioned in this paragraph converge.

Introducing non-homogeneous coordinates by the definitions

$$(16)\qquad x = x_2/x_1 , \qquad y = x_3/x_1 , \qquad z = x_4/x_1 ,$$

we find at once, by use of (15), the following expansions:

$$(17)\quad \begin{cases} x = \Delta u + (\theta_u \Delta u^2 + \gamma \Delta v^2)/2 + [(-2p + \theta_u^2 + \theta_{uu})\Delta u^3 + 3(\beta\gamma + \theta_{uv})\Delta u^2 \Delta v \\ \qquad + 3\gamma\varphi \Delta u \Delta v^2 + (\gamma_v + \gamma\theta_v)\Delta v^3]/6 + \cdots , \\ y = \Delta v + (\beta \Delta u^2 + \theta_v \Delta v^2)/2 + [(\beta_u + \beta\theta_u)\Delta u^3 + 3\beta\psi \Delta u^2 \Delta v \\ \qquad + 3(\beta\gamma + \theta_{uv})\Delta u \Delta v^2 + (-2q + \theta_v^2 + \theta_{vv})\Delta v^3]/6 + \cdots , \\ z = \Delta u \Delta v + (\beta \Delta u^3 + 3\theta_u \Delta u^2 \Delta v + 3\theta_v \Delta u \Delta v^2 + \gamma \Delta v^3)/6 + [(\beta_u + \beta\theta_u)\Delta u^4 \\ \qquad + 2(\pi - 3p + \theta_u^2 + \theta_{uu})\Delta u^3 \Delta v + 3(2\theta_{uv} + \theta_u\theta_v + \beta\gamma)\Delta u^2 \Delta v^2 \\ \qquad + 2(\chi - 3q + \theta_v^2 + \theta_{vv})\Delta u \Delta v^3 + (\gamma_v + \gamma\theta_v)\Delta v^4]/12 + \cdots . \end{cases}$$

From the expansions (17) it is possible to compute an expansion for z as a power series in x, y by setting z equal to a power series in x, y with undetermined coefficients and then demanding that the expansions for x, y, z in

(17) shall satisfy this equation identically in Δu, Δv as far as the terms of any desired order. Thus we find* to terms of the fourth order

$$(18)\quad \begin{cases} z=xy-(\beta x^3+\gamma y^3)/3+(\beta\varphi x^4-4\beta\psi x^3y-6\theta_{uv}x^2y^2-4\gamma\varphi xy^3 \\ \qquad\qquad\qquad\qquad\qquad\qquad +\gamma\psi y^4)/12+\cdots. \end{cases}$$

By way of application of this expansion let us study the intersection of the surface S and the tangent plane, $z=0$, at a point P_x of S. This intersection is a plane curve C. Referred to the triangle x, x_u, x_v the equation of the curve C is the right member of equation (18) set equal to zero. Inspecting the terms of lowest degree therein we see that *the tangent plane at an ordinary point of a surface intersects the surface in a curve with a node at the point, the nodal tangents being the asymptotic tangents of the surface at the point. This node is ordinarily a simple double point.* To demonstrate the truth of the last statement let us expand y as a power series in x along the branch of the curve C which is tangent to the u-tangent, $z=y=0$. The result is

$$y=\beta x^2/3-\beta\varphi x^3/12+\cdots.$$

Therefore this branch of the curve C ordinarily has only two consecutive points on the u-tangent. A similar argument holds for the other branch of C.

Let us drop for a moment the restriction $\theta=\log\beta\gamma$ which is characteristic of Fubini's canonical form of the fundamental differential equations. The necessary calculations would show that $\beta\varphi$ is to be replaced by $2\beta\theta_u-\beta_u$ in the coefficient of x^4 in the expansion (18). When β is not identically zero, the conditional equation $\beta=0$ ordinarily defines a curve on the surface S. It is immediately seen that every point P_x on this curve is ordinarily an inflexion point on that branch of the plane curve C (of intersection of S and the tangent plane at P_x) which is tangent to the u-tangent at P_x. At such a point the node on the curve C is therefore a *flecnode,* i.e., both an inflexion and a node. A point P_x on the surface S where the curve C possesses this singularity is called a *flecnode point,* or sometimes simply a flecnode. The curve $\beta=0$ is called a *flecnode curve,* because every one of its points is a flecnode point. Similarly, with u and v interchanged, the curve $\gamma=0$ is also a flecnode curve on the surface S. By equation (18) we now easily prove the following theorem, which the foregoing discussion suggests:

An asymptotic tangent ordinarily intersects a surface S in three coincident points at its point of contact. But at a point of the flecnode curve $\beta=0$, the asymptotic u-tangent, $z=y=0$, intersects S in four coincident points; similarly for the v-tangent and the curve $\gamma=0$.

* Lane, 1927. 10.

In order to connect the definition of flecnode points formulated in the last paragraph with the definition of flecnode points stated in Section 13, let us consider the canonical form of equations (II, 20) for which $p_{ik}=0$ $(i, k=1, 2)$. It is easy now to verify that the x defined by equation (II, 14) satisfies a system of equations of the form (1); in fact this system is

$$\begin{aligned} &x_{uu}=0 \ , \\ &x_{vv}=-(q_{22}+uq_{12})x+[u^2q_{12}+u(q_{22}-q_{11})-q_{21}]x_u \ , \end{aligned}$$

in which v is the t of Chapter II. Therefore *the flecnode curves on a ruled surface, as defined in Section* 13, *are identical with the curve* $\gamma=0$ *when* β *is identically zero, in the notation of the present section.* The tangent plane at a point of a flecnode curve on a ruled surface cuts the surface in the generator through the point and in a curve with an inflexion at the point. The entire intersection consequently has a node as well as an inflexion at the point, which is customarily and quite correctly called a flecnode. But since every point on a straight line is an inflexion point, a flecnode on a ruled surface is more precisely a *biflecnode*, that is, a node which is an inflexion point on both branches of the curve.

Any plane, except the tangent plane, containing an ordinary asymptotic tangent at a point of a surface intersects the surface in a curve having an inflexion at the point. To prove this statement let us write the equation of a plane through the u-tangent, $z=y=0$, at a point P_x of the surface S in the form

$$z=hy \qquad (h\neq 0) \ .$$

The equation of the cone with its vertex at the point x_{uv} and containing the curve C' of intersection of the surface S and this plane is found, by eliminating z, to be

$$hy=xy-(\beta x^3+\gamma y^3)/3+\cdots \ .$$

Solving this equation for y as a power series in x, and then cutting this cone by the tangent plane, $z=0$, we obtain the equations of a curve,

$$z=0 \ , \qquad y=-\beta x^3/3h+\cdots \ .$$

This curve in the tangent plane has an inflexion at the point P_x when β is not zero, and consequently so does the curve C', as was to be proved. For this reason the asymptotic tangents of a surface are sometimes called *inflexional tangents.*

18. Quadrics of Darboux. There is a three-parameter family of quadric surfaces having contact of the second order at a point of an analytic surface, and among these quadrics there is a one-parameter family called *the quadrics of Darboux* which are distinguished from the others by a certain property which will be explained presently. Although common usage associates the name of Darboux with these quadrics, it seems that they were first considered by Hermite. Among the quadrics of Darboux there are some covariant quadrics of particular interest. Perhaps the most important of these are *the quadric of Lie* and *the quadric of Wilczynski*. Bompiani and Klobouček have independently defined certain generalizations of the quadric of Lie which are called by Bompiani *asymptotic osculating quadrics* and which it seems appropriate to discuss in this section, although they are not themselves quadrics of Darboux.

It is useful to state here a few pertinent facts from the general theory of the contact of an algebraic surface and an analytic surface. *An algebraic surface A is said to have contact of order k with an analytic surface S, at an ordinary point P_x on S, in case every curve on the surface S through the point P_x has at P_x exactly $k+1$ consecutive points on the surface A.* If then the series (15) are substituted in the local algebraic equation of the surface A, this equation is satisfied identically in Δu, Δv as far as the terms of degree k. *It is therefore $(k+1)(k+2)/2$ conditions for an algebraic surface to have contact of order k at a point of an analytic surface.* If an algebraic surface A has contact of order k with an analytic surface S at a point P_x, then A intersects S in a curve* with a multiple point of order $k+1$ at P_x, whose tangents lie in the directions obtained by setting equal to zero the terms of degree $k+1$ that result when the series (15) are substituted in the equation of the surface A and the increments Δu, Δv are replaced by the differentials du, dv.

Since it is known that *an algebraic surface of order h depends*† *on $h(h^2+6h+11)/6$ parameters,* there is a limitation on the order of contact which it is possible for a given algebraic surface to have; this is expressed by the inequality $(k+1)(k+2)/2 \lesseqgtr h(h^2+6h+11)/6$. For example, a plane is of order one and depends on three parameters, so that the tangent plane, $x_4=0$, at a point P_x of a surface S is uniquely determined by the three conditions that it have contact of the first order with S at P_x.

A quadric surface, being of order two, depends on nine parameters. Since contact of the second order imposes six conditions, it follows that *there are ∞^3 quadrics having contact of the second order with a surface S at a point P_x.* The equation of a general one of these is obtained by writing the equation

* Chasles, 1870. 1, p. 354. Letter from Moutard to Poncelet dated 1863.

† Snyder and Sisam, 1914. 1, p. 206.

of the most general non-singular quadric and demanding that the series (15) satisfy this equation identically in Δu, Δv as far as the terms of the second degree. The result can be written in the form

$$x_2x_3+x_4(-x_1+k_2x_2+k_3x_3+k_4x_4)=0\ , \tag{19}$$

where the coefficients k_2, k_3, k_4 are arbitrary, being constants as long as the point P_x is fixed, and functions of u, v when P_x varies over S. This quadric cuts the surface S in a curve with a triple point at P_x, whose tangents are in the directions satisfying the equation

$$\beta du^3+3k_2du^2dv+3k_3dudv^2+\gamma dv^3=0\ . \tag{20}$$

There are certain curves called *the curves of Darboux* and others called *the curves of Segre,* which play a very significant rôle in the geometry of surfaces in ordinary space, and which will now be introduced. It is a simple matter to verify that if the binary cubic differential form that appears in equation (20) is a perfect cube of a linear form, then this form is one of the three linear factors of the cubic form $\beta du^3+\gamma dv^3$, and moreover k_2, k_3 have one of the three pairs of values given by

$$k_2=\epsilon(\beta^2\gamma)^{1/3}\ ,\qquad k_3=\epsilon^2(\beta\gamma^2)^{1/3}\qquad\qquad(\epsilon^3=1)\ . \tag{21}$$

A geometrical consequence is that *if the three directions that satisfy equation* (20) *coincide at all, they coincide in one of the triple of directions satisfying the equation*

$$\beta du^3+\gamma dv^3=0\ . \tag{22}$$

These are by definition *the directions* of Darboux,* and the tangents in these directions are *the tangents of Darboux,* which were called by him *tangentes d'osculation quadrique.* The curves defined by the curvilinear differential equation (22) on the surface S are called *the curves of Darboux,* being enveloped by the tangents of Darboux. A characteristic geometrical property of these curves may be stated as follows:

The curves of Darboux on a surface S in ordinary space are the three one-parameter families of curves such that at each point P_x of S the three tangents of these curves are in the directions in which may coincide the three triple-point tangents of the curve of intersection of S and a quadric having second-order contact with S at P_x.

* Darboux, 1880. 2, p. 356.

The directions of Segre* at a point P_x of a surface S in ordinary space are defined to be the harmonic conjugates of the directions of Darboux with respect to the asymptotic directions. *The tangents of Segre* are by definition the tangents in the directions of Segre. *The curves of Segre* are defined to be the curves enveloped by the tangents of Segre, and their differential equation is

$$\beta du^3 - \gamma dv^3 = 0 \ . \tag{23}$$

The quadrics of Darboux, which it is the principal purpose of the present section to study, can be introduced in the following way. Comparison of equations (22), (20) shows that any quadric (19) with $k_2 = k_3 = 0$ and with k_4 arbitrary cuts the surface S in a curve whose triple-point tangents are the tangents of Darboux at the point P_x. *These* ∞^1 *quadrics, represented by the equation*

$$x_2x_3 - x_1x_4 + k_4x_4^2 = 0 \qquad (k_4 \text{ arbitrary}) \ ,$$

are called the quadrics of Darboux at the point P_x of the surface S. A characteristic geometric property of these quadrics may be stated as follows:

At a point P_x of a surface S in ordinary space the quadrics of Darboux are the ∞^1 quadrics each of which has second-order contact with S at P_x and cuts S in a curve whose triple-point tangents are the tangents of Darboux of S at P_x.

Since contact of the third order imposes ten conditions, and since a quadric surface depends on nine parameters, it would seem possible to satisfy these conditions at particular points on a surface S. The method of counting constants suggests that it would be sufficient to impose one condition on the parameters u, v; then there would be a curve of points at each of which a quadric having contact of the third order would exist. But this conclusion is not† correct. If the surface S is unspecialized, equation (20) shows that a point where there exists a quadric having contact of the third order must have curvilinear coordinates satisfying the two conditions $\beta = \gamma = 0$, and the quadric at such a point is subject only to the restrictions $k_2 = k_3 = 0$. Thus the following conclusions are justified:

A point on an analytic surface S at which there exists a quadric surface having contact of the third order is an intersection of the two flecnode curves on the surface S. At every such intersection the ∞^1 quadrics of Darboux have contact of the third order with the surface S.

Several covariant quadrics of Darboux have been defined by various geometers. One of the most important of these quadrics is *the quadric of*

* Segre, 1908. 1, p. 410.

† Hermite, 1873. 1, p. 149.

Lie, which is defined as follows. *From one of the families of asymptotic curves on a surface S let us select a curve C, and on C a point P_x. At P_x and two neighboring points P_1, P_2 on C let us construct the tangents of the asymptotic curves of the other family. These three asymptotic tangents determine a quadric surface. The limit of this quadric, as P_1, P_2 independently approach P_x along the curve C, is* the quadric of Lie at the point P_x of the surface S.* The equation of the quadric of Lie will show that it is a quadric of Darboux and that it remains the same if the rôles of the two families of asymptotic curves are interchanged in its definition. The u-tangent at a point P_x is determined (see Fig. 10) by x, x_u; and the u-tangent at a point X, near P_x and on the asymptotic v-curve through P_x, is determined by X, X_u represented by the expansions

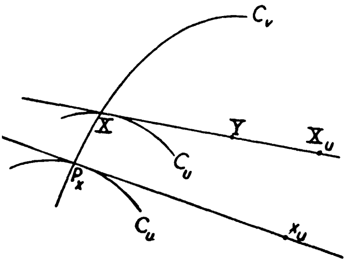

FIG. 10

$$\begin{aligned} X &= x+x_v\Delta v+x_{vv}\Delta v^2/2+\cdots\,, \\ X_u &= x_u+x_{uv}\Delta v+x_{uvv}\Delta v^2/2+\cdots\,. \end{aligned}$$

Any point Y on the latter tangent can be defined by a linear combination of the form

$$Y=hX+kX_u \qquad (h,\ k \text{ scalars})\,. \tag{24}$$

We find, on making use of equations (1), (7), that Y can be expressed in the form

$$Y=x_1x+x_2x_u+x_3x_v+x_4x_{uv}\,,$$

where the local coordinates $x_1, \cdots, x_4$ of the point Y are given by

$$\begin{cases} x_1=h+\cdots\,, \qquad x_2=k+\cdots\,, \\ x_3=h\Delta v+[h\theta_v+k(\beta\gamma+\theta_{uv})]\Delta v^2/2+\cdots\,, \\ x_4=k\Delta v+k\theta_v\Delta v^2/2+\cdots\,. \end{cases} \tag{25}$$

Demanding that the equation of a general quadric be satisfied by the series (25) for $x_1, \ldots, x_4$ identically in h, k, and identically in Δv as far as the

* Lie, 1922. 1, p. 718. Letter from Lie to Klein dated December 18, 1878. Wilczynski, 1908. 2, p. 83.

terms of the second degree, we obtain *the equation of the quadric of Lie* at the point P_x of the surface S, namely,

$$2(x_2x_3-x_1x_4)-(\beta\gamma+\theta_{uv})x_4^2=0\,. \tag{26}$$

That the quadric of Lie is covariant to the surface S, or is independent of the analytic representation used, is evident from its geometric definition. Moreover, this fact can be demonstrated analytically by showing by means of equations (12), (13) that equation (26) is absolutely invariant under the transformation (11).

We shall next consider two quadrics associated with a point of a curve on a surface in ordinary space, which we shall denote by Q_u and Q_v respectively, and which we shall call, with Bompiani, *asymptotic osculating quadrics. The asymptotic osculating quadric Q_u at a point P_x of a curve C on a surface S is* defined to be the limit of the quadric determined by three asymptotic u-tangents constructed at P_x and at two neighboring points of C as these two points independently approach the point P_x along the curve C.* If C is the asymptotic v-curve through the point P_x, then the quadric Q_u is the quadric of Lie, and in this sense Q_u is a generalization of the quadric of Lie. The second asymptotic osculating quadric Q_v can be defined similarly.

In order to find the equation of the quadric Q_u, let us regard the curve C as imbedded in the one-parameter family of curves defined on the surface S by the equation

$$dv-\lambda du=0\,, \tag{27}$$

where λ is a function of u, v. The u-tangent at a point X near the point P_x on the curve C is determined by X, X_u represented by the expansions

$$\begin{aligned} X\ &=x+(x_u+x_v\lambda)\Delta u+(x_{uu}+2x_{uv}\lambda+x_{vv}\lambda^2+x_v\lambda')\Delta u^2/2+\cdots\,,\\ X_u&=x_u+(x_{uu}+x_{uv}\lambda)\Delta u+(x_{uuu}+2x_{uuv}\lambda+x_{uvv}\lambda^2+x_{uv}\lambda')\Delta u^2/2+\cdots\,, \end{aligned}$$

where

$$\lambda'=\lambda_u+\lambda\lambda_v\,.$$

Any point Y on this tangent is defined, as before, by an equation of the form (24), but now we find for the local coordinates $x_1, \ldots, x_4$ of the point Y the expansions

$$\begin{cases} x_1=h+kp\Delta u+\cdots\,, \qquad x_2=k+(h+k\theta_u)\Delta u+\cdots\,,\\ x_3=(h\lambda+k\beta)\Delta u+\{h(\beta+\theta_v\lambda^2+\lambda')+k[\beta_u+\beta\theta_u+2\pi\lambda\\ \qquad\qquad\qquad\qquad\qquad\qquad +(\beta\gamma+\theta_{uv})\lambda^2]\}\Delta u^2/2+\cdots\,,\\ x_4=k\lambda\Delta u+[2h\lambda+k(\beta+2\theta_u\lambda+\theta_v\lambda^2+\lambda')]\Delta u^2/2+\cdots\,. \end{cases} \tag{28}$$

* Bompiani, 1926. 2, p. 263; Klobouček, 1926. 11, p. 342.

Demanding that the equation of a general quadric be satisfied by the series (28) for $x_1, \ldots, x_4$ identically in h, k, and identically in Δu as far as the terms of the second degree, we obtain *the equation of the quadric* Q_u,

$$(29)\left\{\begin{aligned}&2\lambda^3(x_2x_3-x_1x_4)+2\beta\lambda x_4(x_3-\lambda x_2)+\{\beta[\lambda'-\beta+(\varphi-\theta_u)\lambda\\&\qquad\qquad-(2\psi-\theta_v)\lambda^2]-(\beta\gamma+\theta_{uv})\lambda^3\}x_4^2=0\ .\end{aligned}\right.$$

This generalization of the quadric of Lie is obviously not itself a quadric of Darboux, but it does have contact of the second order with the surface S at the point P_x.

The equation of the quadric Q_v can be obtained in a way similar to the foregoing, or else can be written immediately by applying to equation (29) the substitution

$$(30)\qquad \begin{pmatrix} u & 2 & \beta & p & \varphi & \lambda & \lambda' \\ v & 3 & \gamma & q & \psi & 1/\lambda & -\lambda'/\lambda^3 \end{pmatrix} ;$$

the result is

$$(31)\left\{\begin{aligned}&2(x_2x_3-x_1x_4)-2\gamma\lambda x_4(x_3-\lambda x_2)+\{\gamma[-\lambda'-\gamma\lambda^3\\&\qquad\qquad+(\psi-\theta_v)\lambda^2-(2\varphi-\theta_u)\lambda]-(\beta\gamma+\theta_{uv})\}x_4^2=0\ .\end{aligned}\right.$$

The osculating plane of the curve C defined by equation (27) at a point P_x of a surface is determined by x, x', x'', where

$$(32)\qquad \left\{\begin{aligned}&x'=x_u+x_v\lambda\ ,\\&x''=x_{uu}+2x_{uv}\lambda+x_{vv}\lambda^2+x_v\lambda'\qquad (\lambda'=\lambda_u+\lambda\lambda_v)\ .\end{aligned}\right.$$

The local equation of this plane is found by a simple calculation to be

$$(33)\qquad 2\lambda(\lambda x_2-x_3)+(\lambda'+\beta-\theta_u\lambda+\theta_v\lambda^2-\gamma\lambda^3)x_4=0\ ;$$

hence its local coordinates ξ are given by

$$(34)\quad \xi_1=0\ ,\qquad \xi_2=2\lambda^2\ ,\qquad \xi_3=-2\lambda\ ,\qquad \xi_4=\lambda'+\beta-\theta_u\lambda+\theta_v\lambda^2-\gamma\lambda^3\ .$$

Another covariant quadric of Darboux is *Wilczynski's canonical quadric*, which was rediscovered by Bompiani in the following way.* If the curve C has an inflexion at the point P_x, i.e., has three consecutive points in its tangent line at P_x, then its osculating plane (33) is indeterminate so that *at the point* P_x we have $\xi_1=\xi_2=\xi_3=\xi_4=0$, and therefore $\lambda=0$, $\lambda'=-\beta$. Consequently the curve C is tangent to the asymptotic u-curve at P_x, and

* Bompiani, 1927. 11, p. 188.

the asymptotic osculating quadric Q_v at the point P_x of the curve C has the equation

$$2(x_2x_3 - x_1x_4) - \theta_{uv}x_4^2 = 0 .$$

This is* *Wilczynski's canonical quadric,* and was defined by him in an entirely different way which we shall discuss in Chapter VIII, Section 60. Geometrically, the two asymptotic curves at a point play symmetric rôles in spite of the fact that the v-curve is not represented by equation (27). The following theorem summarizes these results:

If a curve on a surface in ordinary space has an inflexion point, it is tangent to an asymptotic curve at the point, and the asymptotic osculating quadric determined by the asymptotic tangents of the other family is the canonical quadric of Wilczynski at the point.

19. Reciprocal congruences. In 1916 G. M. Green called attention to a relation which may exist between two rectilinear congruences whose generators are in correspondence with the points of a given surface. He called

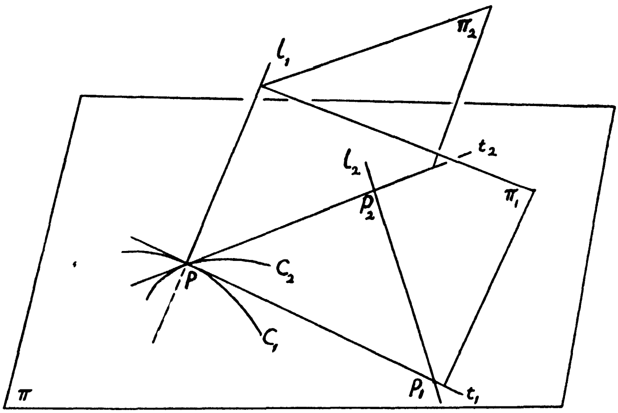

FIG. 11

this relation *the relation R.* It can be defined as follows. Let us consider an unspecialized net N of curves on a surface S. Through a point P of the surface S there pass two curves C_1, C_2 of the net N, with tangents t_1, t_2 at the point P, as drawn in Figure 11. The tangents to the curves of the family of the net N that contains the curve C_1, constructed at the points of the curve C_2, form a non-developable ruled surface R_1; similarly a ruled surface

* Wilczynski, 1908. 2, p. 112.

R_2 can be circumscribed about the surface S along the curve C_1. Let us construct two planes π_1, π_2 through the point P, distinct from the tangent plane π of S at P, and containing respectively the tangents t_1, t_2. These planes intersect in a line l_1 which passes through the point P but does not lie in the tangent plane π. The plane π_1 touches the ruled surface R_1 at a point P_1, and similarly π_2 is tangent to R_2 at a point P_2. The line l_2 joining P_1, P_2 lies in the tangent plane π but does not pass through the point P. *The two lines l_1, l_2 are said to be in the relation R with respect to the net N.* As the point P varies over the surface S, the lines l_1, l_2 generate two congruences Γ_1, Γ_2 which are also said to be in the relation R with respect to the net N.

When the net N is the asymptotic net on the surface S, the lines l_1, l_2 are reciprocal polar lines with respect to the quadric of Lie at the point P of the surface S (see Ex. 34). This means that the polar planes of two and hence all points on the line l_1 with respect to the quadric of Lie contain l_2, and vice versa. Then the lines l_1, l_2, and likewise the congruences Γ_1, Γ_2, are simply said to be *reciprocal* with respect to the surface S. The first problem is to determine the developables and focal surfaces of two reciprocal congruences, in order to prepare the way for the study of particular pairs of reciprocal congruences in the next section.

Let us consider an integral surface S of equations (6), with the parametric vector equation $x=x(u, v)$; a point P_x on S; and *any line l_1 through P_x but not in the tangent plane of S at P_x.* Such a line may be regarded* as determined by the point P_x and the point P_y defined by

$$y=-ax_u-bx_v+x_{uv}\ , \tag{35}$$

where the coefficients a, b are scalar functions of u, v. *As u, v vary, the line l_1 generates a congruence Γ_1.*

We shall immediately prove that *the lines of the congruence Γ_1 can be assembled into a one-parameter family of developable surfaces in two ways (ordinarily determinate and distinct), so that there is one developable of each family through each line l_1.* We shall also show that *the lines of the congruence Γ_1 are the common tangents of two surfaces (called focal surfaces of the congruence), each of the surfaces being the locus of one focus of the variable generator l_1.* These properties of the congruence Γ_1 are properties that are possessed by every rectilinear congruence, due account being taken of limiting cases.

Let us consider a curve C, defined by the parametric equations $u=u(t)$, $v=v(t)$, through a point P_x on a surface S, as shown in Figure 12. If the

* Green, 1919. 1, p. 86.

ruled surface of the congruence Γ_1 that intersects the surface S in the curve C is a developable, and if the point P_z defined by the formula

$$z = y + \lambda x \qquad (\lambda \text{ scalar}) \tag{36}$$

is the corresponding focal point of the line l_1, then l_1 is tangent to the locus

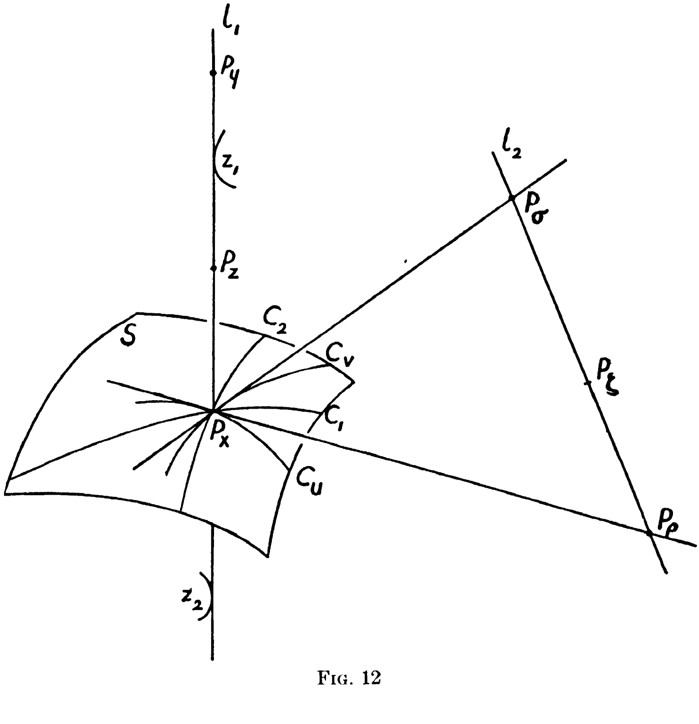

FIG. 12

of the point P_z as t varies, so that z' may be expressed as a linear combination of x and y only. We find at once

$$z' = z_u u' + z_v v' \ ,$$

and after some calculation

$$\begin{aligned} z_u &= (p_v - ap + \beta q + \lambda_u)x + (A + \lambda)x_u + (F - 2a\beta + \beta\psi)x_v - (b - \theta_u)y \ , \\ z_v &= (q_u - bq + \gamma p + \lambda_v)x + (G - 2b\gamma + \gamma\varphi)x_u + (B + \lambda)x_v - (a - \theta_v)y \ , \end{aligned}$$

where the functions A, B, F, G are defined by

$$(37) \qquad \begin{cases} A=-a_u-ab+\beta\gamma+\theta_{uv}\ , & F=p-b_u+b\theta_u-b^2+a\beta\ , \\ B=-b_v-ab+\beta\gamma+\theta_{uv}\ , & G=q-a_v+a\theta_v-a^2+b\gamma\ . \end{cases}$$

Setting equal to zero the coefficients of x_u and x_v in the expression thus obtained for z' as a linear combination of x, x_u, x_v, y, we have

$$(38) \qquad \begin{cases} (A+\lambda)du+(G-2b\gamma+\gamma\varphi)dv=0\ , \\ (F-2a\beta+\beta\psi)du+(B+\lambda)dv=0\ . \end{cases}$$

Eliminating λ we obtain *the differential equation of the curves in which the developables of the congruence* Γ_1 *intersect the surface* S, *which will be called briefly the* Γ_1*-curves of the congruence:*

$$(39) \qquad (F-2a\beta+\beta\psi)du^2-(b_v-a_u)dudv-(G-2b\gamma+\gamma\varphi)dv^2=0\ .$$

The form of this equation shows that the Γ_1-curves ordinarily form a net, and that the lines of the congruence Γ_1 can then be assembled into ∞^1 developables in two distinct ways, so that there are two different developables of Γ_1 containing each line l_1. Moreover, eliminating the ratio dv/du from equations (38) we obtain

$$(40) \qquad \lambda^2+(A+B)\lambda+AB-(F-2a\beta+\beta\psi)(G-2b\gamma+\gamma\varphi)=0\ .$$

If λ_1, λ_2 are the roots of this equation, the corresponding points z_1, z_2 given by the formula (36) are the focal points of the lines l_1. The locus of each of the points z_1, z_2 is a focal surface of the congruence Γ_1. The line l_1 is tangent to both of these surfaces, its points of contact being its two focal points. The reader may complete the discussion in the case in which the discriminant of equation (39) vanishes, and the case in which this equation is indeterminate, showing that in the first case the congruence Γ_1 consists of one family of the asymptotic tangents of a surface, and that in the second case the congruence Γ_1 is a bundle of lines.

We now start with a congruence Γ_1 and arrive at a reciprocal congruence Γ_2 by a geometrical construction. Let us consider a line l_1 determined by a point P_x and the corresponding point P_y. The polar plane of the point P_x with respect to the quadric of Lie (26) is the tangent plane, $x_4=0$; the polar plane of the point P_y has the equation

$$x_1+bx_2+ax_3+(\beta\gamma+\theta_{uv})x_4=0\ .$$

These two planes intersect in a line l_2 which has the equations

$$x_4 = x_1 + bx_2 + ax_3 = 0 \, , \tag{41}$$

and which crosses the asymptotic tangents at the points with local coordinates $(-b, 1, 0, 0)$, $(-a, 0, 1, 0)$. In the original coordinate system these are the points P_ρ, P_σ given by

$$\rho = x_u - bx \, , \qquad \sigma = x_v - ax \, . \tag{42}$$

Clearly, *the line l_2 lies in the tangent plane and does not pass through the point P_x*. As u, v vary, the line l_2 generates a congruence Γ_2 of lines some one of which lies in each tangent plane of the surface S but does not pass through the contact point. Conversely, the local equations of the polar planes of the points P_ρ, P_σ with respect to the quadric of Lie are respectively

$$x_3 + bx_4 = 0 \, , \qquad x_2 + ax_4 = 0 \, . \tag{43}$$

These two planes intersect in the line l_1 with which we started, so that l_1, l_2 are reciprocal polar lines with respect to the quadric of Lie (or any quadric of Darboux), and Γ_1, Γ_2 are in this sense reciprocal congruences. It is clear that *if either of the congruences Γ_1 and Γ_2 is given, the other is uniquely determined,* so that it is possible to start with either congruence and from it arrive at the other by a geometrical construction.

We shall now find the developables and focal surfaces of the congruence Γ_2. If, as the point P_x describes a curve C on the surface S, the line l_2 generates a developable of the congruence Γ_2, and if the point P_ζ defined by the formula (see Fig. 12)

$$\zeta = \rho + \mu\sigma \qquad (\mu \text{ scalar}) \tag{44}$$

is the corresponding focal point of the line l_2, then l_2 is tangent to the locus of the point P_ζ, so that ζ' may be expressed as a linear combination of ρ and σ only. We find

$$\zeta' = (\rho_u + \mu_u\sigma + \mu\sigma_u)u' + (\rho_v + \mu_v\sigma + \mu\sigma_v)v' \, ,$$

where

$$\rho_u = Fx - (b - \theta_u)\rho + \beta\sigma \, , \qquad \rho_v = -(b_v + ab)x - b\sigma + x_{uv} \, ,$$

$$\sigma_v = Gx + \gamma\rho - (a - \theta_v)\sigma \, , \qquad \sigma_u = -(a_u + ab)x - a\rho + x_{uv} \, .$$

Setting equal to zero the coefficients of x and x_{uv} in the expression thus obtained for ζ' as a linear combination of x, ρ, σ, x_{uv}, we have

$$\left\{\begin{aligned} &[F-\mu(a_u+ab)]du+[-b_v-ab+\mu G]dv=0\ ,\\ &\mu du+dv=0\ . \end{aligned}\right. \tag{45}$$

Elimination of μ gives *the differential equation of the curves on the surface S corresponding to the developables of the congruence Γ_2, which will be called the Γ_2-curves of the congruence:*

$$Fdu^2-(b_v-a_u)dudv-Gdv^2=0\ . \tag{46}$$

Moreover, elimination of the ratio dv/du gives

$$F+(b_v-a_u)\mu-G\mu^2=0\ . \tag{47}$$

If μ_1, μ_2, are the roots of this equation, the corresponding points ζ_1, ζ_2, given by the formula (44) are the focal points of the line l_2. Discussion of the special cases is left to the reader.

We often find it convenient to use the locution *a line* l_1 at a point of a surface to mean a line that passes through the point but does not lie in the tangent plane of the surface at the point. Similarly, *a line* l_2 at a point of a surface will mean a line that lies in the tangent plane at the point but does not itself pass through the point. The expressions *a congruence* Γ_1 and *a congruence* Γ_2 are likewise used generically. The connotation of the subscripts 1 and 2 in this connection should be fixed in mind.

Conjugate tangents and conjugate nets will now be defined. *At a point of a surface in ordinary space two tangents are said to be conjugate, or to lie in conjugate directions, in case they separate the asymptotic tangents harmonically. A net of curves on such a surface is said to be a conjugate net in case the two tangents of the curves of the net at each point of the surface are conjugate tangents.* It is not difficult to show that *when the asymptotic net is parametric, conjugate directions dv/du have opposite signs.*

We conclude with a few remarks connecting the relation of conjugacy just now introduced with the theory of reciprocal congruences. The second of equations (45) shows that *the conjugate of a tangent of a Γ_2-curve passes through the corresponding focal point of the line l_2.* Inspection of equations (39) and (46) makes it evident that, in the ordinary case, *the Γ_1-curves and also the Γ_2-curves of two reciprocal congruences form conjugate nets if, and only if, $b_v-a_u=0$.* A congruence Γ_1 whose developables intersect the fundamental surface S in a conjugate net is said to be *conjugate* to S, and a congruence Γ_2 whose developables correspond to a conjugate net on the surface

S is said to be *harmonic* to S. If a congruence Γ_1 is conjugate to the surface S, the reciprocal congruence Γ_2 is harmonic to S, and conversely.

20. The canonical pencils. Several covariant pairs of reciprocal congruences have been defined by various geometers in independent investigations of the geometry of surfaces. Among the covariant lines that have thus been associated with a point P of a surface S, perhaps the most interesting are *the directrices* of Wilczynski, *the axes* of Čech, *the edges* of Green, and *the projective normal* of Green and Fubini. It is indeed remarkable that all of these lines just mentioned that pass through the point P lie in a flat pencil. This is called *the first canonical pencil* of the surface S at the point P. The reciprocal lines also lie in a flat pencil called *the second canonical pencil* of the surface S corresponding to the point P. These pencils will be studied in this section.

The history of the discovery of the projective normal is interesting. It is clear that the ordinary metric normal at a point of a surface is a line l_1. The important rôle played by this normal in the metric differential geometry of the surface suggested that it would be advantageous to have a projectively defined substitute for this line. Moreover, it would be desirable to retain the property that the developables of the normal congruence intersect the surface in a conjugate net, and any other projective property that the metric normal might have. In 1916 Green announced* the discovery of the line now called *the projective normal*, which he called *the pseudo-normal*. His definition of this line is purely geometric, and his characterization is essentially that which will be explained later on in this section. In 1918 Fubini published† his independent investigations on this subject. He arrived at the projective normal by a very elegant property which associates this line with the extremals of a projectively invariant integral, after the manner in which the metric normal is associated with the extremals of the arc-length integral, namely, the geodesics. This property will be explained in more detail in Section 22. Moreover, the property of the projective normal that it is analytically the simplest covariant line l_1 which generates a congruence Γ_1 that is conjugate to the surface S is also due to Fubini.

The theory which we are now developing of the projective differential geometry of a surface in ordinary space may be observed to be *self-dual*, in the sense that it is invariant under the dualistic correspondence that converts each point of the surface into the tangent plane of the surface at the point (see Ex. 13). Reciprocal congruences correspond to each other in this duality.

* Green, 1916. 4, p. 73; 1917. 2, p. 590; 1919. 1, p. 126.

† Fubini, 1918. 1, p. 1038.

Comparison of equations (39) and (46) shows that, *provided* $b_v - a_u \neq 0$, *the* Γ_1-*curves of a congruence coincide with the* Γ_2-*curves of the reciprocal congruence if, and only if,*

$$a = \psi/2 , \qquad b = \varphi/2 . \tag{48}$$

The two reciprocal congruences thus defined, except in the case $b_v - a_u = 0$, are called *the directrix congruences* of Wilczynski. The lines l_1, l_2 which generate them are denoted by d_1, d_2 respectively, and are called *the directrices* of Wilczynski. The property used* by Wilczynski in defining these lines is that introduced in the next paragraph but one, and has the advantage of not requiring the restriction $b_v - a_u \neq 0$.

We interpolate here a few remarks about line geometry preparatory to the demonstration to be given in the next paragraph. *A linear congruence* by definition consists of the lines that intersect two skew straight lines, which are called *the directrices* of the congruence. *A special linear complex* by definition consists of the lines that intersect one straight line, which is called *the axis* of the complex. It is left to the reader to demonstrate that *the linear complex* (I, 44) *is special in case the coefficients* a_{ik} *in the equation of the complex satisfy the condition*

$$a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} = 0 ,$$

and that *then the coefficients* a_{ik} *are the coordinates of the axis of the complex.* Moreover, two linear complexes intersect in a linear congruence. In fact, the two complexes determine a pencil of linear complexes all of which contain the congruence. The reader may show that in the pencil there are two special linear complexes whose axes are the directrices of the linear congruence. These ideas will be immediately developed in more detail in a special case. A more extensive presentation of the elements of line geometry is found in Chapter VII of Wilczynski's book.†

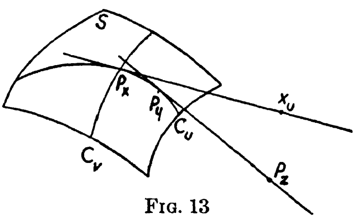

FIG. 13

We shall now prove that *the directrices* d_1, d_2 *of Wilczynski at a point of a surface are the directrices of the linear congruence of intersection of the osculating linear complexes of the two asymptotic curves at the point.* Let us consider the asymptotic curve C_u through a point P_x on a surface S, and let us consider also a point P_y on C_u near P_x, as illustrated in Figure 13. The local

* Wilczynski, 1908. 2, p. 114.

† Wilczynski, 1906. 1.

coordinates $y_1, \ldots, y_4$ of the point P_y are found, by expanding x in powers of Δu and expressing $x_{uu}, \ldots, x_{uuuuu}$ as linear combinations of x, x_u, x_v, x_{uv}, to be given by the power series

$$(49)\quad \begin{cases} y_1 = 1 + p\Delta u^2/2 + \cdots, \\ y_2 = \Delta u + \theta_u \Delta u^2/2 + (p + \theta_u^2 + \theta_{uu})\Delta u^3/6 + \cdots, \\ y_3 = \beta\Delta u^2/2 + (\beta_u + \beta\theta_u)\Delta u^3/6 + [(\beta_u + \beta\theta_u)_u + \beta(p + \theta_u^2 + \theta_{uu}) \\ \qquad\qquad + \beta\pi]\Delta u^4/24 + \cdots, \\ y_4 = \beta\Delta u^3/6 + (\beta_u + \beta\theta_u)\Delta u^4/12 + [3(\beta_u + \beta\theta_u)_u + 2\theta_u(\beta_u + \beta\theta_u) \\ \qquad\qquad + \beta(p + \theta_u^2 + \theta_{uu}) + \beta\pi]\Delta u^5/120 + \cdots. \end{cases}$$

The local coordinates $z_1, \ldots, z_4$ of a point P_z on the tangent of the curve C_u at the point P_y are found similarly, by expanding x_u in powers of Δu, to be given by certain series which turn out to be precisely the derivatives of the series (49) with respect to Δu. Therefore the line coordinates ω_{ik} of the tangent of the curve C_u at the point P_y, as these coordinates were defined in equation (I, 42), are given by the series

$$(50)\quad \begin{cases} \omega_{12} = 1 + \cdots, \qquad \omega_{13} = \beta\Delta u + \cdots, \\ \omega_{14} = \beta\Delta u^2/2 + (\beta_u + \beta\theta_u)\Delta u^3/3 + [3(\beta_u + \beta\theta_u)_u + 2\theta_u(\beta_u + \beta\theta_u) \\ \qquad\qquad \beta(3p + \pi + \theta_u^2 + \theta_{uu})]\Delta u^4/24 + \cdots, \\ \omega_{23} = \omega_{14} + \beta^2\psi\Delta u^4/12 + \cdots, \\ \omega_{42} = -\beta\Delta u^3/3 + \cdots, \qquad \omega_{34} = \beta^2\Delta u^4/12 + \cdots. \end{cases}$$

Demanding that the equation (I, 44) of a linear complex be satisfied by the series (50) for ω_{ik} identically in Δu as far as the terms in Δu^4, we obtain *the equation of the osculating linear complex at the point P_x of the asymptotic curve C_u*, namely,

$$(51)\qquad \omega_{23} - \omega_{14} - \psi\omega_{34} = 0\ .$$

Similarly, or by the substitution (30), the equation of *the osculating linear complex of the asymptotic curve C_v at P_x* is found to be

$$(52)\qquad \omega_{23} + \omega_{14} - \varphi\omega_{42} = 0\ .$$

All of the lines common to the two complexes (51), (52) are also in every complex of the pencil

$$h(\omega_{23} - \omega_{14} - \psi\omega_{34}) + k(\omega_{23} + \omega_{14} - \varphi\omega_{42}) = 0\ ,$$

where h, k are homogeneous parameters. In this pencil there are two special complexes; for them we find that $h = \pm k$, so that their equations are respectively

$$(53) \qquad 2\omega_{23} - \varphi\omega_{42} - \psi\omega_{34} = 0 , \qquad 2\omega_{14} - \varphi\omega_{42} + \psi\omega_{34} = 0 .$$

The two osculating complexes (51), (52) have in common all the lines of the linear congruence whose directrices are the axes of the special complexes (53). To obtain the equations in point coordinates of each of these directrices it is sufficient to write by means of the equations (I, 45) the conditions that must be satisfied by the coordinates of a point if the plane corresponding to it in the null system of each of the special complexes (53) is indeterminate. In this way, after a little calculation which will be omitted, we obtain again the directrices of Wilczynski, thus completing the proof.

Let us pass to the definition of another pair of covariant reciprocal lines. It can be shown that *at a point P_x of a surface the osculating planes of the three curves of Segre intersect in a line l_1 for which*

$$(54) \qquad a = \psi/3 , \qquad b = \varphi/3 .$$

To make the demonstration one replaces λ' by $\lambda_u + \lambda\lambda_v$ in equation (33). One then replaces λ by $\omega\lambda$ and again by $\omega^2\lambda$, where ω is a complex cube root of unity. Thus one has three equations, in which the direction λ of a curve of Segre has the value $(\beta/\gamma)^{1/3}$, as equation (23) shows. Finally one takes suitable linear combinations of the three equations and reduces the result by means of the characteristic condition $\theta = \log \beta\gamma$ and the definitions of φ, ψ given in equations (9). These calculations are straightforward and elementary, and the details will be omitted. This line* of intersection is now commonly called *the first axis* of Čech, although he himself called it *the line of Segre*. It and its reciprocal, *the second axis*, will be denoted by a_1, a_2 respectively.

The directrix d_1 and the axis a_1 at a point P_x on a surface S determine a plane through P_x called† *the canonical plane* of the surface S at the point P_x. The equation of this plane, shown in Figure 14, is

$$(55) \qquad \varphi x_2 - \psi x_3 = 0 .$$

The pencil of lines lying in this plane and having its center at P_x is called *the first canonical pencil* of the surface S at the point P_x, and any line of this pencil, joining $P_x(1, 0, 0, 0)$ to a point

$$(0, k\psi, k\varphi, 1) \qquad (k = \text{const.}) ,$$

* Čech, 1922. 2, p. 199. † Fubini and Čech, 1926. 1, p. 155.

is spoken of as *a canonical line of the first kind.* The canonical plane (55) intersects the tangent plane, $x_4=0$, in a line called *the first canonical tangent* t_1 of S at P_x.

The directrix d_2 and the axis a_2 which lie in the tangent plane at the point P_x of the surface S intersect in a point called *the canonical point* of the surface S *corresponding* to the point P_x, or briefly *at* P_x. The local coordinates of this point are $0, \psi, -\varphi, 0$. The pencil of lines lying in the tangent plane and

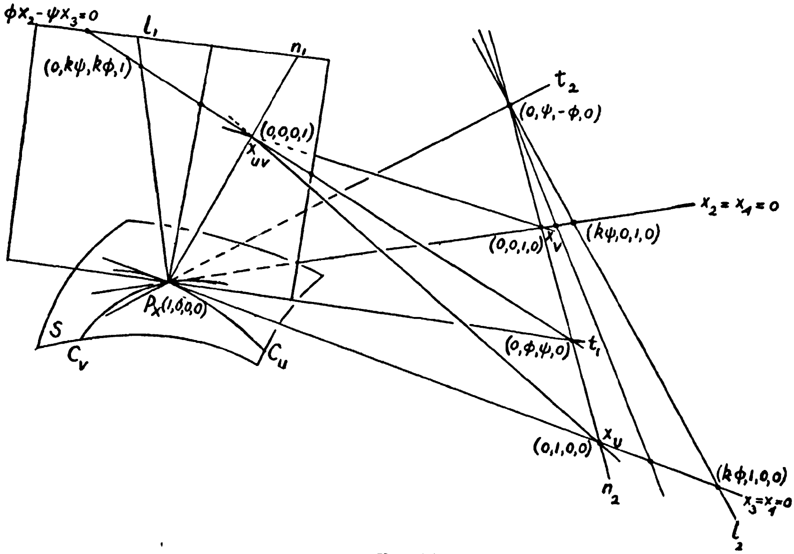

FIG. 14

having its center at the canonical point is called *the second canonical pencil* of the surface S corresponding to the point P_x, and any line of this pencil, crossing the asymptotic tangents at the points $(k\varphi, 1, 0, 0)$ and $(k\psi, 0, 1, 0)$ is spoken of as *a canonical line of the second kind. The second canonical tangent* t_2 by definition joins the point P_x to the canonical point. It should be observed that *the two canonical tangents are conjugate tangents.* It should be further observed that *a canonical line of the first kind and the canonical line of the second kind with the same value of k are a pair of reciprocal lines for which*

$$a=-k\psi\,, \qquad b=-k\varphi \qquad (k=\text{const.})\,. \tag{56}$$

For *the directrices* of Wilczynski we have seen that $k=-1/2$, and for *the axes* of Čech, $k=-1/3$. We shall now consider* *the edges* e_1, e_2 of Green,

* Green, 1916. 4, p. 73; 1917. 2, p. 591; 1919. 1, p. 114.

which may be defined analytically as those canonical lines for which $k=-1/4$; and *the projective normal* n_1 of Green and Fubini, which may be defined analytically as the canonical line l_1 for which $k=0$. Finally, we shall show how any canonical line can be characterized by a certain cross ratio.

It will first be shown, after the manner of Green's investigations, that *the edges* have the following characteristic geometric property. *At each point P_x of a surface S the edge e_2 crosses the tangent of each asymptotic curve in the pole of the other asymptotic tangent with respect to the four-point conics of the projection of the first asymptotic curve from any point on the edge e_1 onto the tangent plane, and that this geometric relation holds for no other pair of reciprocal lines.* For the purpose of the proof let us make the following observations. If the coordinates of a point referred to the tetrahedron x, x_u, x_v, x_{uv} are $y_1, \ldots, y_4$, and if the coordinates of the same point are $x_1, \ldots, x_4$ when referred to the tetrahedron $x, \rho, \sigma, y+\lambda x$, where ρ, σ are defined by equations (42) and y by (35), and λ is an arbitrary scalar function of u, v, then the identity

$$y_1x+y_2x_u+y_3x_v+y_4x_{uv}=x_1x+x_2\rho+x_3\sigma+x_4(y+\lambda x)$$

yields the equations for the *transformation of coordinates* between the two tetrahedrons; after solution for $x_1, \ldots, x_4$ these equations can be written in the form

$$\tag{57} \begin{cases} x_1=y_1+by_2+ay_3+(2ab-\lambda)y_4\,, \\ x_2=y_2+ay_4\,, \qquad x_3=y_3+by_4\,, \\ x_4=y_4\,. \end{cases}$$

The parametric equations of the projection C_u' of the asymptotic curve C_u from the new vertex (0, 0, 0, 1) onto the tangent plane, $x_4=0$, are found by substituting the series (49) for $y_1, \ldots, y_4$ into equations (57) and taking such a linear combination of the resulting coordinates $x_1, \ldots, x_4$ and of 0, 0, 0, 1 as will make the fourth coordinate vanish. These parametric equations, to terms of as high degree as will be needed, are

$$\tag{58} \begin{cases} x_1=1+b\Delta u+\cdots\,, \\ x_2=\Delta u+\theta_u\Delta u^2/2+\cdots\,, \\ x_3=\beta\Delta u^2/2+(\beta_u+\beta\theta_u+b\beta)\Delta u^3/6+\cdots\,, \end{cases}$$

the fourth coordinate being zero in the remainder of this paragraph. It will be observed that equations (58) are the result of substituting the series (49) for $y_1, \ldots, y_4$ into the first three of equations (57) and neglecting the fourth equation. It will be observed further that, as far as written, these equations

are independent of λ and hence of the position of the center of projection on the line l_1 that is now being used as the edge $x_2=x_3=0$ of the tetrahedron of reference. Imposing on the general equation of a conic the conditions that it be satisfied by the series (58) for x_1, x_2, x_3 identically in Δu as far as the terms in Δu^3, we obtain *the equation of the four-point conics at the point P_x of the projection C_u' of the asymptotic curve C_u*, namely,

$$6x_1x_3-3\beta x_2^2+2(\varphi-4b)x_2x_3+hx_3^2=0\ , \tag{59}$$

where h is a parameter. The polar line of the point P_ρ (0, 1, 0) with respect to any one of these conics (see Fig. 15) has the equation

$$3\beta x_2-(\varphi-4b)x_3=0\ ,$$

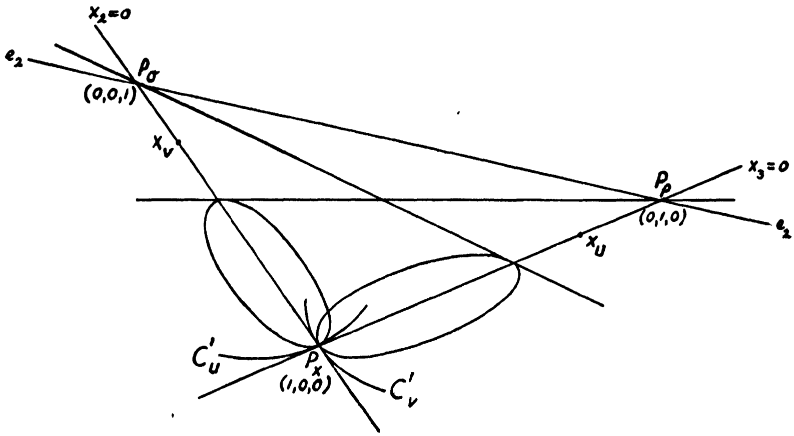

FIG. 15

and this line coincides with the asymptotic v-tangent, $x_2=0$, in case $b=\varphi/4$. Similarly, with the rôles of the asymptotics C_u and C_v interchanged, we get $a=\psi/4$. Thus we obtain the edges of Green, as was to be proved.

We next establish the following theorem, which states a characteristic geometric property of the projective normal defined analytically as the canonical line for which $k=0$. *The projective normal n_1 is the harmonic conjugate of the directrix d_1 with respect to the edge e_1 and the canonical tangent t_1.* This statement is immediately verified by observing that the cross ratio of four canonical lines is the cross ratio of the corresponding values of the constant k, and that

$$(e_1t_1d_1n_1)=(-1/4,\ \infty,\ -1/2,\ 0)=-1\ .$$

Green used essentially this property to define the projective normal.

We are now prepared to complete the geometric description of the local tetrahedron introduced in Section 17 at a point P_x of a surface. We observe that *the line* xx_{uv} *is the projective normal,* and that *the line* $x_u x_v$ *is its reciprocal* n_2. Thus we have the geometric definition of two more lines, besides the asymptotic tangents, of the tetrahedron, which was already known to be covariant. The vertices x_u, x_v are the intersection points of the asymptotic tangents and the line n_2. The lines $x_u x_{uv}$, $x_v x_{uv}$ are respectively the v-tangent and u-tangent of the surfaces generated by the points x_u, x_v.

Setting $a=b=0$ in Section 19 we find that the Γ_1-curves of the projective normal congruence, which are called *the projective lines of curvature,* have the equation

$$\pi du^2 - \chi dv^2 = 0 \,, \tag{60}$$

while the corresponding Γ_2-curves have the equation

$$p du^2 - q dv^2 = 0 \,. \tag{61}$$

Both of these nets of curves are conjugate nets.

The cross ratio of the canonical tangent t_1, the projective normal n_1, the directrix d_1, and any canonical line c_1, which corresponds to a general constant k, has the value $-2k$. Thus the general canonical line c_1 can be characterized by its cross ratio with three other lines each of which has already been characterized geometrically.

21. Conjugate nets. The historical reason why conjugate tangents at a point of a surface are so named is that they can be defined as conjugate diameters of the Dupin indicatrix* of the surface at the point. Instead of this metric definition a projective definition of conjugate tangents was stated in the latter part of Section 19.

It is known that there are infinitely many conjugate nets, as defined in Section 19, on a non-developable surface in ordinary space. In fact, when any one-parameter family of non-asymptotic curves is given on such a surface, there exists another one-parameter family of curves on the surface such that the two families form a conjugate net. When it is a matter of studying only a single conjugate net, it may be advantageous to choose this net for the parametric net, as will be done in the next chapter. But in solving problems concerning several, or infinitely many, conjugate nets on a surface, it is convenient to choose the asymptotic net on the surface as the parametric net.

* Dupin, 1813. 1, p. 41.

When a conjugate net is given on a surface in ordinary space, there are two congruences of the types Γ_1 and Γ_2 determined by the net in a way that will be described presently and called respectively *the axis congruence* and *the ray congruence* of the net. These are of fundamental importance for the geometry of the net. Moreover, there is a one-parameter family of conjugate nets determined in a certain way by the given net on the surface and called a *pencil of conjugate nets.* At each point of the surface there are several loci associated with such a pencil. Among these, two of the most significant are a certain cubic curve in the tangent plane, called *the ray-point cubic,* and its dual called *the axis-plane* cone. These configurations will be studied in this section.

The curvilinear differential equation of a conjugate net N_λ on an integral surface S of system (1) can be written in the form

$$dv^2-\lambda^2du^2=0\ , \tag{62}$$

where λ is a function of u, v. There exists on the surface S a conjugate net uniquely determined by the net N_λ, with the property that at each point of S its tangents separate the tangents of the net N_λ harmonically. This net is called *the associate conjugate net* of the net N_λ, and its differential equation is easily shown to be

$$dv^2+\lambda^2du^2=0\ . \tag{63}$$

In fact, direct calculation verifies that the two tangents of the net (62) and the two tangents of the net (63) form a harmonic group, so that the relation between the two nets is entirely reciprocal.

The axis congruence of a conjugate net is determined by the net in the following way. *The axis* of a point P_x on a surface S with respect to a conjugate net N_λ on S, is defined to be the line of intersection of the two osculating planes at P_x of the two curves of the net N_λ that pass through the point P_x.* This line is also called *the axis of the net N_λ at the point P_x.* Referring to Figure 16, let us denote the two curves of the net N_λ that pass through the point P_x by C_λ and $C_{-\lambda}$ according as the direction dv/du has the value λ or $-\lambda$. At the point P_x the osculating plane of the curve C_λ has the equation (33), while the osculating plane of the curve $C_{-\lambda}$ has the equation obtained therefrom by first replacing λ' by $\lambda_u+\lambda\lambda_v$ and then changing the sign of λ. Taking suitable linear combinations of these two equations, we obtain two equations of the form (43) with a, b given by

$$a=[\theta_v+(\log\lambda)_v+\beta/\lambda^2]/2\ ,\qquad b=[\theta_u-(\log\lambda)_u+\gamma\lambda^2]/2\ . \tag{64}$$

* Wilczynski, 1915. 1, p. 312.

Therefore *the axes of all the points of a surface S with respect to a conjugate net N_λ on S form a congruence* Γ_1; this is called *the axis congruence* of the net N_λ.

The associate axis congruence of a conjugate net is by definition the axis congruence of the associate conjugate net. The associate axis congruence of the net N_λ is also a congruence Γ_1, and for it the values of a, b are obtained from equations (64) by changing the sign of λ^2.

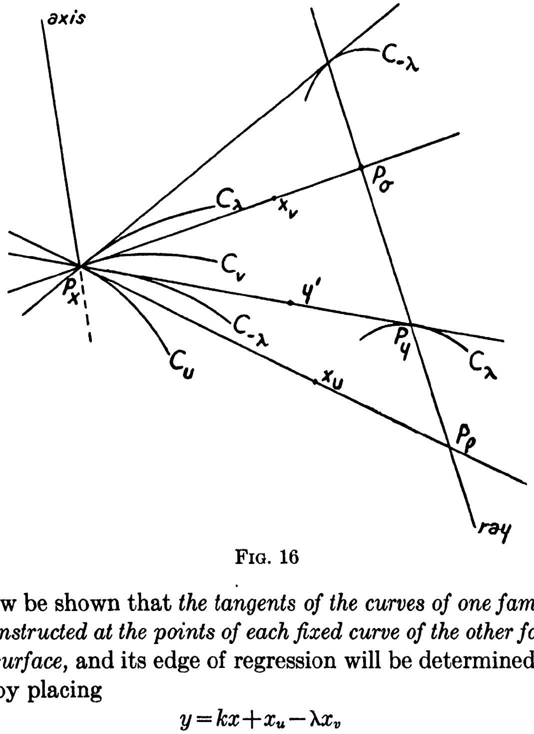

FIG. 16

It will now be shown that *the tangents of the curves of one family of a conjugate net constructed at the points of each fixed curve of the other family form a developable surface,* and its edge of regression will be determined. The point P_y defined by placing

$$y = kx + x_u - \lambda x_v \qquad (k \text{ scalar})$$

is on the tangent of the curve $C_{-\lambda}$ at the point P_x. As the point P_x varies along the curve C_λ, the point P_y generates a curve whose tangent at P_y is determined by P_y and the point y' given by

$$y' = y_u + \lambda y_v = (p + k_u + k_v\lambda - q\lambda^2)x + (\theta_u + k - \gamma\lambda^2)x_u + (\beta - \lambda' + k\lambda - \theta_v\lambda^2)x_v \, .$$

This point is on the tangent of the curve $C_{-\lambda}$ in case k satisfies the equation

$$(65) \qquad 2\lambda k - \lambda' + \beta + \theta_u\lambda - \theta_v\lambda^2 - \gamma\lambda^3 = 0 \, .$$

Therefore, as the point P_x varies along the curve C_λ, the tangent of the curve $C_{-\lambda}$ generates a developable whose edge of regression is generated by the focal point of the tangent of the curve $C_{-\lambda}$, namely, the point

$$(66) \qquad (\lambda'-\beta-\theta_u\lambda+\theta_v\lambda^2+\gamma\lambda^3)x+2\lambda(x_u-\lambda x_v) \; .$$

A similar argument can be made with the curves C_λ and $C_{-\lambda}$ interchanged.

The ray congruence of a conjugate net is determined by the net in the following way. The point (66) is called *the ray-point* of the curve C_λ, corresponding to the point P_x, or sometimes simply the ray-point of C_λ *at* P_x. Its local coordinates are given by

$$(67) \qquad \begin{cases} x_1=\lambda'-\beta-\theta_u\lambda+\theta_v\lambda^2+\gamma\lambda^3 \; , \\ x_2=2\lambda \; , \qquad x_3=-2\lambda^2 \; , \qquad x_4=0 \; . \end{cases}$$

The ray-point of the curve $C_{-\lambda}$, corresponding to the same point P_x, is obtained by changing the sign of λ in equations (67). The line joining these two points is called *the ray* of the point P_x with respect to the net N_λ*, or sometimes also *the ray of the net N_λ corresponding to the point P_x*, or simply the ray of N_λ *at* P_x. Taking linear combinations so as to eliminate first x_v and then x_u from the expression (66) and the expression obtained therefrom by changing the sign of λ, we find that the ray crosses the asymptotic tangents at the points (42) with a, b given by

$$(68) \qquad a=[\theta_v+(\log\lambda)_v-\beta/\lambda^2]/2 \; , \qquad b=[\theta_u-(\log\lambda)_u+\gamma\lambda^2]/2 \; .$$

Therefore *the rays of all the points of a surface S with respect to a conjugate net N_λ on S form a congruence Γ_2*; this is called *the ray congruence* of the net N_λ. Comparison of equations (64) and (68) shows that *the ray congruence of a conjugate net is the reciprocal of the associate axis congruence of the net.* Similarly, defining *the associate ray congruence* of a conjugate net to be the ray congruence of the associate conjugate net, we see that *the axis congruence of a conjugate net is the reciprocal of the associate ray congruence of the net.*

We now define a pencil of conjugate nets. *The class of ∞^1 conjugate nets on a surface every one of which has the property that at every point of the surface its two tangents form with the tangents of a fundamental conjugate net the same cross ratio is called a pencil† of conjugate nets.* The differential equation of a general net $N_{\lambda h}$ of the pencil p_λ of conjugate nets determined by the net N_λ whose equation is (62) can be written in the form

$$(69) \qquad dv^2-\lambda^2h^2du^2=0 \qquad (h=\text{const.}) \; ,$$

* *Ibid.*, p. 317.

† Wilczynski, 1920. 1, p. 216.

since the cross ratio of the four values of dv/du calculated from equations (62) and (69) is constant.

The curve called *the ray-point cubic* next claims our attention. In order to define this curve let us consider a point P_x on a surface S, a pencil p_λ of conjugate nets on S, a net $N_{\lambda h}$ of this pencil, and the curve $C_{\lambda h}$ of this net that passes through the point P_x. The local coordinates of the ray-point, corresponding to the point P_x, of the curve $C_{\lambda h}$ are found from equations (67), on replacing therein λ by λh, to be given by

$$x_1 = \lambda_u h + \lambda\lambda_v h^2 - \beta - \theta_u \lambda h + \theta_v \lambda^2 h^2 + \gamma\lambda^3 h^3 ,$$
$$x_2 = 2\lambda h , \qquad x_3 = -2\lambda^2 h^2 , \qquad x_4 = 0 .$$

Homogeneous elimination of h from these equations gives the equations of the locus of this ray-point as the net $N_{\lambda h}$ varies over the pencil p_λ, namely,

$$x_4 = l x_2 x_3 - \beta x_2^3 - \gamma x_3^3 = 0 , \tag{70}$$

where l is defined by

$$l = 2x_1 + [\theta_u - (\log \lambda)_u] x_2 + [\theta_v + (\log \lambda)_v] x_3 . \tag{71}$$

The locus of the ray-point, corresponding to the point P_x, of the curve $C_{-\lambda h}$ of the net $N_{\lambda h}$ that passes through P_x is found, by the foregoing argument with the sign of h changed, to be the same curve (70). This curve of the third order in the tangent plane of the surface S at the point P_x is * called *the ray-point cubic* of the pencil p_λ, corresponding to the point P_x. It has a double point at P_x with the asymptotic tangents at P_x for double point tangents (see Fig. 17). It has three points of inflexion which lie on the line $x_4 = l = 0$, called *the flex-ray* of the pencil p_λ corresponding to the point P_x, some one of the inflexions lying on each of the lines

$$x_4 = \beta x_2^3 + \gamma x_3^3 = 0 . \tag{72}$$

These three lines are *the tangents of Darboux*, as can be seen by referring to equation (22) and by observing that the local equations of the tangent to a curve $dv - \lambda du = 0$ at P_x are $x_4 = x_3 - \lambda x_2 = 0$. Comparison of the equations $x_4 = l = 0$ and (41) shows that *the flex-rays of a pencil p_λ, corresponding to all the points of a surface, form a congruence Γ_2 for which*

$$a = [\theta_v + (\log \lambda)_v]/2 , \qquad b = [\theta_u - (\log \lambda)_u]/2 . \tag{73}$$

* Lane, 1922. 3, p. 289.

The reader may have observed that when the asymptotics are real there is only one real tangent of Darboux at each point P_x, and likewise only one real tangent of Segre, and only one real inflexion of the ray-point cubic (70). We may remark further in this connection that the envelope of the rays

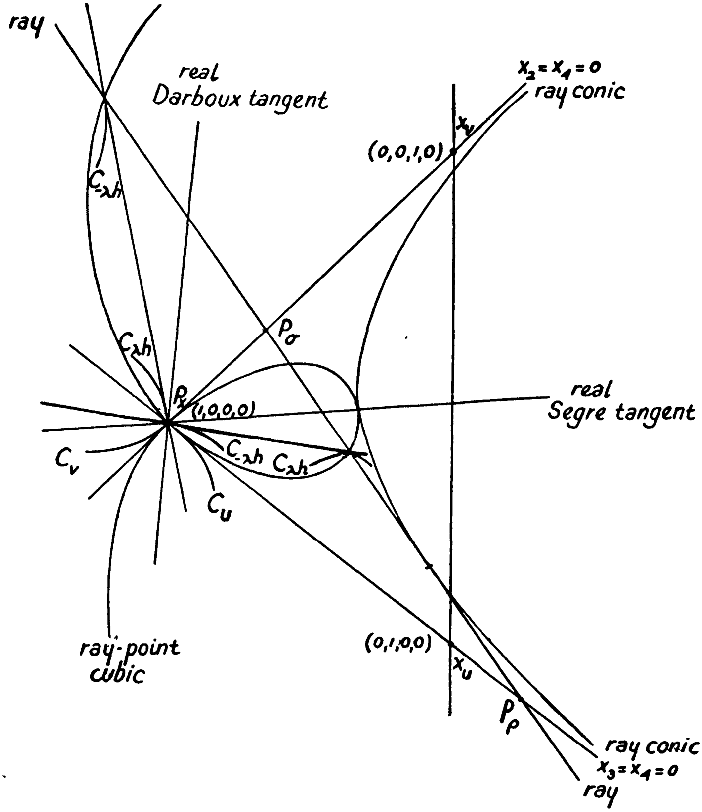

FIG. 17

of all the nets of a pencil p_λ, corresponding to a point P_x, is a conic called *the ray conic* (see Exs. 11, 25) which has interesting relations to the ray-point cubic.

The dual of the ray-point cubic of a pencil of conjugate nets, corresponding to a point of a surface, is called *the axis-plane cone* of the pencil at the point. The axis-plane cone of the pencil p_λ at the point P_x of the surface S is, therefore, the envelope of the osculating planes at the point P_x of all the

curves of the pencil p_λ that pass through P_x. The local coordinates ξ of the osculating plane at the point P_x of the curve $C_{\lambda h}$ can be obtained from equations (34) by replacing therein λ by λh. Then homogeneous elimination of h gives the equations of the envelope of this osculating plane as the net $N_{\lambda h}$ varies over the pencil p_λ, namely,

$$\xi_1 = m\xi_3\xi_2 - \beta\xi_3^3 - \gamma\xi_2^3 = 0 \; , \tag{74}$$

where m is defined by

$$m = 2\xi_4 - [\theta_u - (\log \lambda)_u]\xi_3 - [\theta_v + (\log \lambda)_v]\xi_2 \; . \tag{75}$$

The same equation (74) would have been obtained if the curve $C_{-\lambda h}$ had been used instead of the curve $C_{\lambda h}$, and this is *the equation of the axis-plane cone* of the pencil p_λ at the point P_x. This cone is of the third class and has its vertex at the point P_x. It has the tangent plane of S at P_x for bitangent plane; in fact, it touches this plane along the asymptotic tangents through P_x. It has three cusp-planes which pass through the line $\xi_1 = m = 0$, called *the cusp-axis of the pencil* p_λ at the point P_x, some one of the planes intersecting the tangent plane in each of the lines

$$\xi_1 = \beta\xi_3^3 + \gamma\xi_2^3 = 0 \; . \tag{76}$$

These three lines are *the tangents of Segre*, as can be seen by observing that in point coordinates the equations of the tangents of Segre are $x_4 = \beta x_2^3 - \gamma x_3^3 = 0$, and that for a line through P_x we have $\xi_1 = 0$, $x_2/\xi_3 = x_3/(-\xi_2)$. *The cusp-axes of a pencil p_λ, at all the points of a surface, form a congruence Γ_1 for which a, b are given by equations* (73), as the equations $\xi_1 = m = 0$ show. Hence we deduce the theorem:

The flex-ray and cusp-axis congruences of a pencil of conjugate nets are reciprocal congruences.

The curves (22) of Darboux and the curves (23) of Segre belong to a pencil of conjugate nets called* *the Segre-Darboux pencil*, for which $\lambda = (\beta/\gamma)^{1/3}$. *The cusp-axis of the Segre-Darboux pencil is the axis a_1 of Čech*, since for this pencil equations (73) reduce to (54). Moreover, the flex-ray of the Segre-Darboux pencil is the second axis a_2 of Čech.

22. Hypergeodesics. Union curves and planar systems. The curves† defined on a surface S by a differential equation of the form

$$v'' = A + Bv' + Cv'^2 + Dv'^3 \; , \tag{77}$$

* *Ibid.*, p. 293.

† Fubini, 1918. 1, p. 1034; Wilczynski, 1922. 4.

in which the coefficients are functions of u, v and accents indicate total differentiation with respect to u, are called *hypergeodesics*, because the differential equation of the geodesics in the metric theory of surfaces can be written in this form (see § 48, Chap. VI). Since (77) is a differential equation of the second order, it follows that this equation defines a *two-parameter family* of hypergeodesics on the surface S. Within a suitably restricted region of S there is just one of these hypergeodesics through any two distinct points. Moreover, a hypergeodesic is uniquely determined by a point on S and a direction at the point. Other properties of hypergeodesics will be discovered presently; and some special kinds of hypergeodesics will be considered, among which are *the curves of a pencil of conjugate nets*, the so-called *projective geodesics* connected with Fubini's definition of the projective normal, the *union curves* of a congruence Γ_1, and *planar systems* of curves.

The equations of *the envelope of the osculating planes at a point P_x of all the hypergeodesics* (77) *that pass through P_x* are found by replacing v' by λ in equation (77), substituting the resulting expression for λ' in equations (34), and eliminating λ homogeneously. The result is

$$\xi_1=\xi_3\xi_2[2\xi_4-(\theta_u-B)\xi_3-(\theta_v+C)\xi_2]-(\beta+A)\xi_3^3-(\gamma-D)\xi_2^3=0\ . \tag{78}$$

This envelope is therefore a cone, which has properties similar to those of the cone (74). In particular it has a cusp-axis, which is called *the cusp-axis of the hypergeodesics* at the point P_x, and which passes through the point $(0, -a, -b, 1)$ for which

$$a=(\theta_v+C)/2\ , \qquad b=(\theta_u-B)/2\ . \tag{79}$$

Dually, *the locus of the ray-points, corresponding to the point P_x, of all the hypergeodesics* (77) *that pass through P_x* is the curve

$$x_4=x_2x_3[2x_1+(\theta_u-B)x_2+(\theta_v+C)x_3]-(\beta-A)x_2^3-(\gamma+D)x_3^3=0\ . \tag{80}$$

This curve is to be compared with the curve (70); it has a line of inflexions which is the reciprocal of the cusp-axis of the cone (78) and is called *the flex-ray of the hypergeodesics* at the point P_x.

Our first projective example of a family of hypergeodesics will now be adduced. If equation (69) is solved for h, and if h is then eliminated by total differentiation with respect to u, it becomes apparent that *the curves of a pencil p_λ of conjugate nets constitute a family of hypergeodesics* for which

$$A=D=0\ , \qquad B=(\log\lambda)_u\ , \qquad C=(\log\lambda)_v\ . \tag{81}$$

The vanishing of the coefficients A, D signifies that the asymptotic curves are included in the family of hypergeodesics. Moreover in this example we obviously have $B_v = C_u$.

In order to obtain another example of hypergeodesics, let us observe that the differential form $\beta\gamma du dv$ is easily shown to be absolutely invariant under the transformation (11). Therefore the integral

$$\int (\beta\gamma v')^{1/2} du$$

calculated along a curve is invariant. This integral is called *the projective arc-length* of the curve along which it is calculated. Making use of Euler's equation for the extremals of an integral of the form $\int \varphi(u, v, v')du$, namely,

$$\varphi_{v'v'} v'' = \varphi_v - \varphi_{uv'} - \varphi_{vv'} v' \;, \tag{82}$$

we find that the differential equation of the extremals of the projective arc-length is of the form (77) with

$$A = D = 0 \;, \qquad B = \theta_u \;, \qquad C = -\theta_v \;. \tag{83}$$

These curves are called *the projective geodesics* on the surface S. Reference to equations (79) makes it evident that *the cusp-axis of the projective geodesics is the projective normal.* This is Fubini's definition* of the projective normal.

Union curves† are a third example of hypergeodesics. *A curve on a surface is called a union curve of a congruence* Γ_1 *in case the curve is such that its osculating plane at each of its points* P_x *contains the line* l_1 *of* Γ_1 *through* P_x. The differential equation of the union curves of a general congruence Γ_1 is obtained by replacing λ by v' in equation (33) and substituting for $x_1, \ldots, x_4$ the coordinates $0, -a, -b, 1$ respectively. The result is an equation of the form (77) with

$$A = -\beta \;, \qquad B = \theta_u - 2b \;, \qquad C = -(\theta_v - 2a) \;, \qquad D = \gamma \;. \tag{84}$$

The envelope of the osculating planes at the point P_x of all these curves that pass through P_x is, of course, the line l_1 of the congruence Γ_1 through P_x, so that these planes form a pencil with l_1 as axis. For this reason the union curves of a congruence are sometimes called‡ *an axial system* of curves.

A curve on a surface S *is called a dual union curve of a congruence* Γ_2 *in case the curve is such that its ray-point corresponding to each of its points* P_x

* Fubini, 1918. 1, p. 1038.

† Sperry, 1918. 2, p. 214.

‡ Bompiani, 1923. 1, p. 268.

lies on the line l_2 of Γ_2 in the tangent plane of S at the point P_x. Using equations (67) with $\lambda = v'$, and the equations (41) of a line l_2, we find that the differential equation of the dual union curves of a congruence Γ_2 is of the form (77) with

$$(85)\qquad A=\beta\ ,\qquad B=\theta_u-2b\ ,\qquad C=-(\theta_v-2a)\ ,\qquad D=-\gamma\ .$$

A planar system of curves* on a surface is defined as follows. Let us consider a congruence Γ of non-singular quadric surfaces one of which is associated with each point P_x of a surface S in such a way that it contains the asymptotic tangents through P_x, but does not have contact of order as high as the second with S at P_x. The equation of such a non-singular quadric can be written in the form

$$(86)\qquad x_2x_3+x_4(k_1x_1+k_2x_2+k_3x_3+k_4x_4)=0\ ,\qquad k_1(1+k_1)\neq 0\ ,$$

in which $k_1, \ldots, k_4$ are constants as long as P_x is fixed, and functions of u, v when P_x varies. Then *a planar curve of the congruence Γ is defined to be a curve on the surface S such that at each of its points P_x its asymptotic osculating quadric of one family* (which is known to contain the asymptotic tangents through the point P_x) *intersects the associated quadric of Γ in a residual pair of straight lines instead of in a residual non-singular conic as would ordinarily be the case.* Using equation (86) and the asymptotic osculating quadric Q_u with equation (29) we find, by eliminating x_1 and setting the discriminant of the result equal to zero, that the differential equation of the corresponding family of planar curves of the congruence Γ is of the form (77) with

$$(87)\qquad \left\{\begin{array}{l} A=r\beta(1-k_1)\ ,\quad B=2rk_2-(\log\gamma)_u\ ,\quad C=-2rk_3+2\psi-\theta_v\ ,\\ \beta D=2(rk_2k_3-k_4)/k_1+\beta\gamma+\theta_{uv}\ ,\qquad r(1+k_1)=1\ .\end{array}\right.$$

The reason why such a family of curves is called *a planar system* will be explained in Exercise 17 of Chapter V.

Using an equation of the form (88) but with coefficients l instead of k, and using the asymptotic osculating quadric Q_v with equation (31), we obtain a second planar system with an equation of the form (77), for which

$$(88)\qquad \left\{\begin{array}{l} \gamma A=-2(sl_2l_3-l_4)/l_1-\beta\gamma-\theta_{uv}\ ,\quad B=2sl_2-2\varphi+\theta_u\ ,\\ C=-2sl_3+(\log\beta)_v\ ,\quad D=-\gamma s(1-l_1)\ ,\qquad s(1+l_1)=1\ .\end{array}\right.$$

* Bompiani, 1926. 2, p. 263.

When the expressions for corresponding coefficients A, B, C, D in equations (87) and (88) are equated, the resulting equations can be solved uniquely for the coefficients l in terms of the coefficients k, and vice versa. Thus the following theorem of Bompiani is established:

If a planar system is defined by one family of asymptotic osculating quadrics and a congruence of quadrics, then a second congruence of quadrics can be determined so that it and the other family of asymptotic osculating quadrics define the same planar system.

23. The transformation of Čech. We begin with a definition. *The transformation* represented analytically by the equations*

$$\left\{\begin{array}{lll} \xi_1=0\ , & \sigma\xi_2=x_2x_3^2\ , & \sigma\xi_3=x_2^2x_3\ , \\ \sigma\xi_4=-x_1x_2x_3+k(\beta x_2^3+\gamma x_3^3) & & \end{array}\right. \qquad (k=\text{const.})\ , \tag{89}$$

where σ is a proportionality factor, between planes with local coordinates ξ through a point P_x of a surface S and points with local coordinates x in the tangent plane of S at P_x, is the transformation of Čech. Observing that $x_4=0$, and solving equations (89) for the ratios of x_1, x_2, x_3, we have the inverse transformation,

$$\left\{\begin{array}{lll} \rho x_1=-\xi_4\xi_3\xi_2+k(\beta\xi_3^3+\gamma\xi_2^3)\ , & & \\ \rho x_2=\xi_3^2\xi_2\ , & \rho x_3=\xi_3\xi_2^2\ , & x_4=0\ , \end{array}\right. \tag{90}$$

where ρ is a proportionality factor. It will be observed that the transformation of Čech is not only a *cubic birational transformation* but is actually a *Cremona transformation.* It will be observed, further, that the transformation involves an arbitrary constant k, which is independent of u, v. By giving particular numerical values to k we see that at each point of a surface there are ∞^1 particular transformations of Čech.

It is of interest to characterize geometrically the general transformation (89) when k is an arbitrary constant, and then to distinguish geometrically any particular one of these transformations from all the rest. This latter distinction is made by first defining geometrically two particular transformations (89), namely, *the polarity of Lie* and *the correspondence of Segre,* and then using a certain cross ratio.

For the purpose of characterizing geometrically the general transformation of Čech, let us consider the pencil of cubic curves

$$x_4=x_2x_3(x_1+bx_2+ax_3)-k(\beta x_2^3+\gamma x_3^3)=0\ , \tag{91}$$

* Čech, 1922. 2, p. 192.

with the same constant k that appears in equations (89), and with arbitrary functions a, b of u, v. This pencil lies in the tangent plane at the point P_x of the surface S. Every cubic of this pencil has a double point at P_x, with the asymptotic tangents through P_x for double-point tangents; and has three inflexions which lie on the line l_2 of equations (41), some one of the inflexions lying on each of the tangents of Darboux (72). These properties are characteristic of the pencil of curves. Now the general transformation of Čech has the following properties. It is a one-to-one correspondence between points in the tangent plane, $x_4=0$, and planes through the contact point $P_x(1, 0, 0, 0)$ such that to every plane $(0, \xi_2, \xi_3, \xi_4)$ through P_x distinct from the tangent plane corresponds a point $(x_1, x_2, x_3, 0)$ on the line $x_4=\xi_2x_2-\xi_3x_3=0$ conjugate to the line of intersection, $x_4=\xi_2x_2+\xi_3x_3=0$, of the given plane and the tangent plane. Moreover, to the planes of a pencil with any line l_1, joining the point $P_x(1, 0, 0, 0,)$ to a point $(0, -a, -b, 1)$, as axis correspond the points on a cubic of the pencil (91), the line of inflexions of the cubic being the line l_2 reciprocal to l_1. Conversely,* every transformation having these properties can be shown to be a transformation of Čech, by a simple argument which will be omitted. Thus the general transformation of Čech is characterized.

Of the several particular transformations of Čech which have received attention at various times in the past, we shall mention here only two. First, the transformation for which $k=0$ is without difficulty recognized to be *the correspondence between points in the tangent plane*, $x_4=0$, *and their polar planes with respect to the quadric of Lie*, represented by equation (26). This transformation between the points and planes of space is sometimes called *the polarity of Lie*. Second, the transformation for which $k=1$ is *the correspondence of Segre*, defined to be *the correspondence between the ray-points corresponding to the point* P_x *and the osculating planes at the point* P_x *of all the curves on the surface* S *that pass through* P_x, as can be verified by eliminating λ, λ' homogeneously between equations (34) and (67).

To construct the point P_k corresponding to a given plane in any particular transformation of Čech with a definitely assigned value of k we may proceed as follows. First draw the line conjugate to the line of intersection of the given plane and the tangent plane. On this conjugate line mark the point P_0 which is the pole of the given plane with respect to the quadric of Lie, and the point P_1 which corresponds to the given plane in the correspondence of Segre. Then the required point P_x can be located by making use of the fact that the cross ratio of the points P_x, P_0, P_1, P_k in the order named is k. This construction distinguishes the particular transformation of Čech from all the rest.

* Lane, 1926. 3, p. 209.

24. Pangeodesics and the cone of Segre. The pangeodesics* on a surface are a covariant system of curves which will be precisely defined presently as extremals of a certain projectively invariant integral. These curves will also be characterized geometrically by a very simple property of their osculating planes and their asymptotic osculating quadrics. The theory of the pangeodesics is closely connected with the theory of the *plane sections* of the surface and the dual curves, namely, the *curves of contact of cones circumscribing the surface.* The envelope of the osculating planes, at a point P_x of a surface S, of all the pangeodesics on S that pass through P_x is called *the cone of Segre* of S at P_x; this cone was defined by Segre in a closely related way, which will be explained later on in this section.

Let us give our attention now to defining the pangeodesics analytically. It is easy to verify that the differential expression $(\beta du^3+\gamma dv^3)/dudv$ is absolutely invariant under the transformation (11), so that the integral

$$\int[(\beta+\gamma v'^3)/v']du \tag{92}$$

calculated along a curve on a surface S is invariant. Euler's equation (75) for the extremals of this integral reduces to

$$2(\beta+\gamma\lambda^3)/\lambda'=\beta_u\lambda+2\beta_v\lambda^2-2\gamma_u\lambda^4-\gamma_v\lambda^5 \tag{93}$$

when v' is replaced by λ. These extremals are called *pangeodesics,* and (93) is *the differential equation of the pangeodesics on the surface* S, in the sense that when λ is a solution of this equation then v is determined as a function of u along a pangeodesic by means of equation (27). It is perhaps worthy of note that equation (93) can be written in the form

$$[\beta-2\gamma\lambda^3)/\lambda^2]_u=[(2\beta-\gamma\lambda^3)/\lambda]_v\ .$$

A very simple characteristic geometric property† of the pangeodesics can be discovered in the following way. The two asymptotic osculating quadrics (29), (31) at a point P_x of a curve C_λ belonging to the family defined by equation (27) intersect in the asymptotic tangents through the point P_x and in a conic which lies in a plane through P_x. The equation of this plane is found, by eliminating x_1 from equations (29) and (31), to be

$$\left\{\begin{aligned}&2\lambda(\beta+\gamma\lambda^3)(x_3-\lambda x_2)+[(\beta+\gamma\lambda^3)(\lambda'-\beta+\gamma\lambda^3)+(\log\gamma)_u\beta\lambda\\&\qquad-(2\psi-\theta_v)\beta\lambda^2+(2\varphi-\theta_u)\gamma\lambda^4-(\log\beta)_v\gamma\lambda^5]x_4=0\ .\end{aligned}\right. \tag{94}$$

* Fubini and Čech, 1926. 1, p. 141.

† Lane, 1927. 3, p. 102.

If $\beta+\gamma\lambda^3=0$, then C_λ is a curve of Darboux. Let us exclude this case. Demanding that the plane (94) coincide with the osculating plane (33) of the curve C_λ we find that λ must be a solution of equation (93), and conversely. Thus we obtain the following characterization of the pangeodesics:

A curve not a curve of Darboux on a surface is a pangeodesic if, and only if, at each of its points its osculating plane contains the conic of intersection of its asymptotic osculating quadrics that does not lie in the tangent plane of the surface at the point.

It is possible to find another characterization of the pangeodesics. With this end in view let us seek for the differential equation of all plane sections of a surface S. A curve C_λ of the family defined by equation (27) on the surface S is a plane curve if, and only if, at every point x on the curve the four points x, x', x'', x''' are coplanar, accents indicating total differentiation with respect to u. Expressing these derivatives as linear combinations of x, x_u, x_v, x_{uv}, and equating to zero the fourth-order determinant of the local coordinates of the four points, we obtain *the differential equation of all the plane curves on the surface S*,

$$(95)\qquad 2\lambda(M'+2\lambda\pi-2\lambda^3\chi)-M(3\lambda'+\beta+\theta_u\lambda+\theta_v\lambda^2+\gamma\lambda^3)=0\ ,$$

where M is defined by

$$(96)\qquad M=\lambda'+\beta-\theta_u\lambda+\theta_v\lambda^2-\gamma\lambda^3\ .$$

When λ is replaced by dv/du equation (95) becomes an equation of the third order for v as a function of u along a plane curve. This fact is consistent with the fact that there are ∞^3 planes in ordinary space.

We interpolate here a few remarks on the relation of plane curves and union curves. Placing $a=b=0$ in equations (84), one easily shows that *the equation of the union curves of the projective normal congruence is $M=0$.* These curves are analogous to the geodesics of metric geometry, which can be defined as the union curves of the normal congruence (see § 48, Chap. VI). Moreover, inspection of equations (60) and (95) shows that *a union curve of the projective normal congruence is a plane curve if, and only if, it is a projective line of curvature.* This is reminiscent of the metric theorem that a geodesic is a plane curve if, and only if, it is a line of curvature. This theorem, by a little calculation which will be omitted, can be extended to read:

A union curve of any congruence Γ_1 is a plane curve if, and only if, it is a Γ_1-curve of the congruence.

Let us now turn to certain dual considerations. *The curves of contact of cones circumscribed about a surface S will be called cone curves.* They are dual to the plane curves on S, in the sense that they correspond to the plane

curves in the dualistic transformation that converts every point P_x of the surface S into the tangent plane of S at P_x (see Ex. 13). Just as three consecutive points of a curve C_λ on S determine an osculating plane of C_λ, so the tangent planes of the surface S at these three points intersect in the corresponding ray-point (66) of C_λ. The tangent planes at all points of C_λ envelop a developable surface whose edge of regression is the locus of the ray-point of C_λ. If this developable is a cone, then the ray-point is fixed as the point P_x varies along the curve C_λ, and consequently the corresponding total derivative with respect to u of the expression (66) is proportional to the expression itself. From this proportionality we obtain *the differential equation of all the cone curves on the surface S,*

$$2\lambda(N'+2\lambda p-2\lambda^3 q)-N(3\lambda'-\beta+\theta_u\lambda+\theta_v\lambda^2-\gamma\lambda^3)=0\ , \tag{97}$$

where N is defined by

$$N=\lambda'-\beta-\theta_u\lambda+\theta_v\lambda^2+\gamma\lambda^3\ . \tag{98}$$

The equation of the dual union curves of the reciprocal of the projective normal congruence is $N=0$.

The following statements concern relations among asymptotics, pangeodesics, plane curves, and cone curves. If C_λ is both a plane curve and a cone curve then λ must be a simultaneous solution of equations (95) and (97). Elimination of λ'' from these two equations leads to equation (93). Therefore, *if a curve is both a plane curve and a cone curve, it is a pangeodesic.* But it is easy to show that on an unspecialized surface S not every pangeodesic is a plane curve. In fact, *the asymptotics on S are pangeodesics,* since $\lambda=0$ is a solution of equation (93). Moreover, *if an asymptotic curve is a plane curve then it is a straight line* since, if $\lambda=0$ is a solution of equation (95), then $\beta=0$. Therefore a surface on which all pangeodesics, and hence the asymptotics, are plane curves is restricted to be a quadric. On a quadric surface every plane curve is also a cone curve, and vice versa, and the pangeodesics are indeterminate.

If, for a particular pair of values of u, v which are the curvilinear coordinates of a point P_x on a surface S, equation (95) is satisfied by a function λ without the equation's being satisfied identically by λ, then the curve C_λ of the family defined by equation (27) through the point P_x has* at P_x what is called *a stationary osculating plane.* At such a point the osculating plane of C_λ intersects C_λ in four consecutive points. A dual definition involving equation (97) can be given for *a stationary ray-point* of the curve C_λ,

* Segre, 1908. 1, p. 409.

corresponding to P_x. If the curve C_λ has at the point P_x both a stationary osculating plane and a stationary ray-point then the positions of the osculating plane and of the ray-point are restricted by the condition (93) so that, when the direction of the curve C_λ at the point P_x is given, the osculating plane and the ray-point are determined. *Those curves on the surface S whose osculating planes and ray-points are everywhere in these restricted positions are the pangeodesics.* This is the second geometrical characterization of the pangeodesics. It may be restated as follows:

The pangeodesics are those curves whose osculating planes at each point of the surface are precisely those planes which are able to serve as the osculating planes of curves each of which has at the point both a stationary osculating plane and a stationary ray-point.

The stationary osculating planes and ray-points of curves at a point of a surface were investigated by Segre. He considered all curves on the surface through the point each of which has at the point both a stationary osculating plane and a stationary ray-point, and showed that the envelope of the planes is a cone of the sixth class, now commonly called *the cone of Segre,* while the locus of the ray-points is a curve of the sixth order, which is dual to the cone. It follows from the foregoing discussion that *the envelope of the osculating planes at a point P_x of all the pangeodesics through P_x is the cone of Segre.* Its equations are found by substituting the expression for λ' given by equation (93) into equations (34) and then eliminating λ homogeneously. The result is

$$
(99) \quad \left\{ \begin{aligned} & 2(\beta\xi_3^3-\gamma\xi_2^3)(\beta\xi_3^3+\theta_u\xi_3^2\xi_2+\theta_v\xi_3\xi_2^2+\gamma\xi_2^3-2\xi_4\xi_3\xi_2)-\beta_u\xi_3^5\xi_2+2\beta_v\xi_3^4\xi_2^2 \\ & \qquad -2\gamma_u\xi_3^2\xi_2^4+\gamma_v\xi_3\xi_2^5=\xi_1=0\,. \end{aligned} \right.
$$

The cone of Segre has its vertex at the point P_x, and has the tangent plane of the surface S at P_x for quintuple plane; in fact, it touches this plane along the asymptotic tangents and the tangents of Darboux through P_x. Dually, *the locus of the ray-points corresponding to the point P_x of all the pangeodesics on the surface S through P_x is Segre's curve of the sixth order in the tangent plane of S at P_x.* Its equations can be found directly, or else can be written immediately by means of the result of Exercise 13.

25. The tetrahedron of Demoulin. Let us consider again the quadric of Lie at a point P_x of a surface S. We propose to study the envelope of this quadric as P_x varies on S. Clearly the surface S is part of the envelope, and it will be found that the quadric of Lie ordinarily touches its envelope, besides at P_x, also in four other points. These four points are the vertices of a tetrahedron called *the tetrahedron of Demoulin* of the surface S, corresponding to the point P_x.

In order to find the points where the quadric of Lie touches its envelope, we shall differentiate the equation (26) of the quadric of Lie partially with respect to u and with respect to v, and then solve the two derived equations simultaneously with the original equation for the ratios of the local coordinates $x_1, \ldots, x_4$. But we observe that the tetrahedron of reference varies with the point P_x, so that we need first of all to deduce some formulas for differentiating the local coordinates of a point referred to the local tetrahedron x, x_u, x_v, x_{uv} at the point P_x. We attack this problem immediately.

The local coordinates of a point X near the point P_x on the surface S are given by the power series (15). The local coordinates y, z, w of the corresponding points X_u, X_v, X_{uv}, near the points x_u, x_v, x_{uv} respectively, are represented to terms of the first order by the following power series:

$$(100)\quad \begin{cases} y_1 = p\Delta u + \cdots, \ y_2 = 1+\theta_u\Delta u + \cdots, \ y_3 = \beta\Delta u + \cdots, \ y_4 = \Delta v + \cdots, \\ z_1 = q\Delta v + \cdots, \ z_2 = \gamma\Delta v + \cdots, \ z_3 = 1+\theta_v\Delta v + \cdots, \ z_4 = \Delta u + \cdots, \\ w_1 = (p_v+\beta q)\Delta u + (q_u+\gamma p)\Delta v + \cdots, \\ w_2 = (\beta\gamma+\theta_{uv})\Delta u + \chi\Delta v + \cdots, \\ w_3 = \pi\Delta u + (\beta\gamma+\theta_{uv})\Delta v + \cdots, \\ w_4 = 1+\theta_u\Delta u + \theta_v\Delta v + \cdots. \end{cases}$$

If a point has coordinates $x_1, \ldots, x_4$ referred to the tetrahedron x, x_u, x_v, x_{uv} at the point P_x, and has coordinates $X_1, \ldots, X_4$ referred to the neighboring tetrahedron X, X_u, X_v, X_{uv}, the equations of the transformation between the two tetrahedrons are found from the identity

$$x_1x + x_2x_u + x_3x_v + x_4x_{uv} = X_1X + X_2X_u + X_3X_v + X_4X_{uv}\ .$$

One substitutes for each of X, X_u, X_v, X_{uv} the equivalent linear expression in x, x_u, x_v, x_{uv}, and then equates corresponding coefficients of x, x_u, x_v, x_{uv}. The resulting equations can be solved for $X_1, \ldots, X_4$ by interchanging $X_1, \ldots, X_4$ and $x_1, \ldots, x_4$ respectively and changing the signs of the increments $\Delta u, \Delta v$. The final result is

$$(101)\quad \begin{cases} X_1 = x_1 - x_2p\Delta u - x_3q\Delta v - x_4[(p_v+\beta q)\Delta u + (q_u+\gamma p)\Delta v] + \cdots, \\ X_2 = -x_1\Delta u + x_2(1-\theta_u\Delta u) - x_3\gamma\Delta v - x_4[(\beta\gamma+\theta_{uv})\Delta u + \chi\Delta v] + \cdots, \\ X_3 = -x_1\Delta v - x_2\beta\Delta u + x_3(1-\theta_v\Delta v) - x_4[\pi\Delta u + (\beta\gamma+\theta_{uv})\Delta v] + \cdots, \\ X_4 = -x_2\Delta v - x_3\Delta u + x_4(1-\theta_u\Delta u - \theta_v\Delta v) + \cdots. \end{cases}$$

Holding $v=\text{const.}$ and taking the limit in these formulas as Δu approaches zero (having first transposed certain terms and then divided by Δu), and then interchanging the rôles of u and v and repeating the process, we obtain the desired *formulas for differentiating the local point coordinates:*

$$
(102)\qquad \begin{cases}
x_{1u}=-px_2-(p_v+\beta q)x_4\,, & x_{1v}=-qx_3-(q_u+\gamma p)x_4\,,\\
x_{2u}=-x_1-\theta_u x_2-(\beta\gamma+\theta_{uv})x_4\,, & x_{2v}=-\gamma x_3-\chi x_4\,,\\
x_{3u}=-\beta x_2-\pi x_4\,, & x_{3v}=-x_1-\theta_v x_3-(\beta\gamma+\theta_{uv})x_4\,,\\
x_{4u}=-x_3-\theta_u x_4\,, & x_{4v}=-x_2-\theta_v x_4\,.
\end{cases}
$$

If equation (26) is differentiated partially with respect to u by means of formulas (102), and if the derived equation is simplified by means of (26) itself and the second of the integrability conditions (10), the result can be written as the first of the following equations:

$$
(103)\qquad \begin{cases}
2\beta x_2^2+2\beta\psi x_2x_4+[(\beta\psi)_v-\beta\gamma\varphi-2\beta q]x_4^2=0\,,\\
2\gamma x_3^2+2\gamma\varphi x_3x_4+[(\gamma\varphi)_u-\beta\gamma\psi-2\gamma p]x_4^2=0\,.
\end{cases}
$$

The second of these equations is the result of differentiating with respect to v and simplifying, and can be written easily by means of the substitution (30). Each of the two equations (103) represents a pair of planes. The equations of these planes can be written in the form

$$
(104)\qquad \begin{cases}
x_2-h_1x_4=0\,, & x_2-h_2x_4=0\,,\\
x_3-k_1x_4=0\,, & x_3-k_2x_4=0\,,
\end{cases}
$$

where h_1, h_2 are the roots of the first of equations (103) regarded as a quadratic in x_2/x_4; and similarly for k_1, k_2. Solution of each equation in the first row of (104) with each equation in the second row gives a contact point of the quadric of Lie with its envelope. Therefore, *as the point P_x varies, the quadric of Lie touches its envelope besides at P_x also at the four points P_{11}, P_{12}, P_{22}, P_{21} whose coordinates are*

$$
(105)\qquad [h_ik_j-(\beta\gamma+\theta_{uv})/2,\ h_i,\ k_j,\ 1]\qquad (i,j=1,2).
$$

These points are the vertices of a tetrahedron called the tetrahedron of Demoulin at the point P_x of the surface S.*

The following properties of the tetrahedron of Demoulin will be stated here without proof. Figure 18 may assist in visualizing them. *The four*

* Demoulin, 1908. 3, p. 494.

lines $P_{11}P_{12}$, $P_{12}P_{22}$, $P_{22}P_{21}$, $P_{21}P_{11}$ *are generators of the quadric of Lie forming a simple skew quadrangle, the first and third sides intersecting the v-tangent through the point* P_x *and the other two sides intersecting the u-tangent. The points in which the lines* $P_{12}P_{22}$ *and* $P_{21}P_{11}$ *intersect the u-tangent are the flecnodes on this tangent regarded as a generator of the ruled surface of u-tangents circumscribing the surface S along the v-curve through the point* P_x*; and*

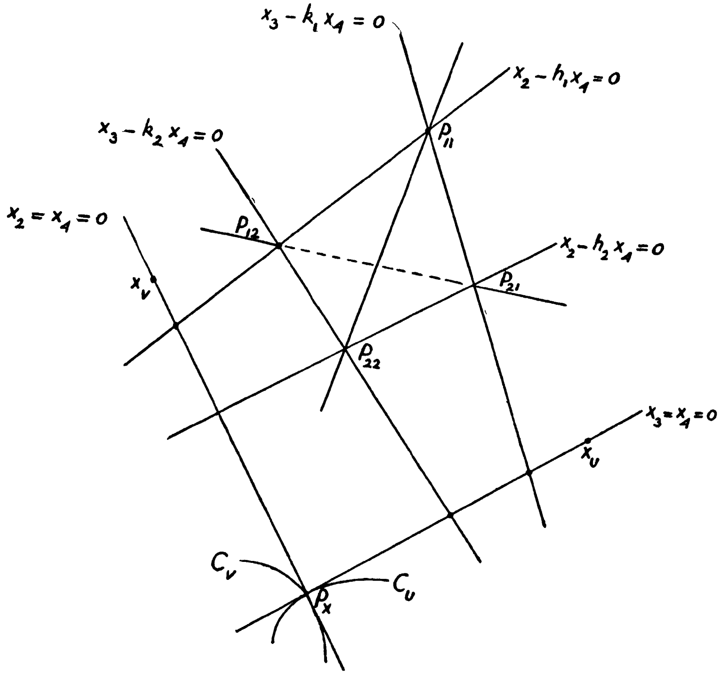

FIG. 18

the lines $P_{12}P_{22}$, $P_{21}P_{11}$ *are the flecnode tangents at these points; similarly with u and v interchanged. The two lines* $P_{11}P_{22}$, $P_{12}P_{21}$ *are polar lines with respect to the quadric of Lie. The two planes* $P_xP_{11}P_{22}$, $P_xP_{12}P_{21}$ *intersect in the first directrix* d_1 *of Wilczynski. The four points* P_{ij} *coincide in case the directrix curves* (i.e., curves corresponding to the developables of the directrix congruences) *coincide with the asymptotic curves.* This subject will be taken up again in Section 41.

Incidentally, we find by means of the formulas (102) the following *formulas for differentiating local line coordinates:*

$$
(106)\qquad \left\{
\begin{aligned}
&\omega_{12u}=-\theta_u\omega_{12}-(\beta\gamma+\theta_{uv})\omega_{14}-(p_v+\beta q)\omega_{42}\ ,\\
&\omega_{13u}=-\beta\omega_{12}-\pi\omega_{14}-p\omega_{23}+(p_v+\beta q)\omega_{34}\ ,\\
&\omega_{14u}=-\omega_{13}-\theta_u\omega_{14}+p\omega_{42}\ ,\\
&\omega_{23u}=-\omega_{13}-\theta_u\omega_{23}+\pi\omega_{42}+(\beta\gamma+\theta_{uv})\omega_{34}\ ,\\
&\omega_{42u}=\omega_{14}+\omega_{23}-2\theta_u\omega_{42}\ ,\\
&\omega_{34u}=\beta\omega_{42}-\theta_u\omega_{34}\ ,\\
&\omega_{12v}=-\gamma\omega_{13}-\chi\omega_{14}+q\omega_{23}-(q_u+\gamma p)\omega_{42}\ ,\\
&\omega_{13v}=-\theta_v\omega_{13}-(\beta\gamma+\theta_{uv})\omega_{14}+(q_u+\gamma p)\omega_{34}\ ,\\
&\omega_{14v}=-\omega_{12}-\theta_v\omega_{14}-q\omega_{34}\ ,\\
&\omega_{23v}=\omega_{12}-\theta_v\omega_{23}+(\beta\gamma+\theta_{uv})\omega_{42}+\chi\omega_{34}\ ,\\
&\omega_{42v}=-\theta_u\omega_{42}+\gamma\omega_{34}\ ,\\
&\omega_{34v}=-\omega_{14}+\omega_{23}-2\theta_v\omega_{34}\ .
\end{aligned}
\right.
$$

EXERCISES

1. A necessary and sufficient condition that a point whose coordinates are functions of u, v be fixed is that the coordinates satisfy two equations of the form $x_u+px=0$, $x_v+qx=0$ with $p_v=q_u$.

2. By means of the transformation (3) with $2\log\lambda=\theta$, reduce system (1) to *Wilczynski's canonical form*,

$$y_{uu}+2by_v+fy=0\ ,\qquad y_{vv}+2a'y_u+gy=0\ .$$

Show that the transformation between Fubini's and Wilczynski's canonical forms is $x=k(\beta\gamma)^{1/2}y$ where $k=\text{const.}$

WILCZYNSKI, 1907. 1, p. 246; FUBINI, 1918. 1, p. 1036

3. Show that by a general transformation of parametric net system (1) takes the form

$$x_{uu}=px+\alpha x_u+\beta x_v+Lx_{uv}\ ,$$
$$x_{vv}=qx+\gamma x_u+\delta x_v+Nx_{uv}\ .$$

Prove further that by a suitable choice of proportionality factor and of independent variables these equations can be reduced to the form $x_{uu}=Lx_{uv}$, $x_{vv}=Nx_{uv}$, for which the parametric curves are plane sections of the surface made by two axial pencils of planes with intersecting axes.

4. If $\beta=\gamma=0$, the integral surfaces of system (1) are all quadrics, since the differential equations can be reduced to the form $x_{uu}=0$, $x_{vv}=0$, of which four particular solutions are given by equations (II, 13).

5. Show by means of the series (18) that the integral surfaces of system (6) are surfaces of the third order in case

$$(\log \beta/\gamma)_{uv}=0\ ,$$
$$4p=\varphi_u-\theta_u\varphi+\varphi^2/4-\beta\psi/3\ ,$$
$$4q=\psi_v-\theta_v\psi+\psi^2/4-\gamma\varphi/3\ .$$

LANE, 1927. 4, p. 475

6. The quadric of Lie at a point P_x of a surface S is the osculating quadric along the generator through P_x of the ruled surface R_u of asymptotic u-tangents constructed at the points of the v-curve through P_x. Obtain the equation (26) of the quadric of Lie as the locus of the tangents of the curved asymptotics, $(\beta\gamma+\theta_{uv})dv+2dw=0$, of R_u at the points x_u+wx of the u-tangents of S through P_x.

7. The quadrics that are tangent to the quadric of Lie at every point of the two asymptotic tangents at a point of a surface are the quadrics of Darboux.

8. If, at a point P_x on a surface S, the curve C_λ of the family defined by equation (27) has the tangent plane of S for stationary osculating plane, then C_λ is tangent to an asymptotic curve, and the asymptotic osculating quadric of C_λ which is determined by three consecutive asymptotic tangents of the other family is a quadric of Fubini,

$$6(x_2x_3-x_1x_4)-(2\beta\gamma+3\theta_{uv})x_4^2=0\ .$$

FUBINI, 1928. 1, p. 16

9. A curve C on a surface is a flecnode curve on the ruled surface of asymptotic u-tangents constructed at the points of C if, and only if, C is a curve of Darboux. In this case C is also a flecnode curve on the ruled surface of asymptotic v-tangents constructed at the points of C. ČECH, 1927. 7, p. 477

10. At a point P_x of a surface the harmonic conjugate of P_x with respect to the two flecnodes on the generator xx_u of the ruled surface R_u (see Ex. 6) lies on the second directrix d_2; similarly with u and v interchanged.

FUBINI and ČECH, 1926. 1, pp. 148 and 226

• 11. The envelope of the rays, corresponding to a point P_x of a surface, of all the conjugate nets of a pencil p_λ is a conic (called *the ray conic* of p_λ corresponding to P_x). Its equations are $x_4=4\beta\gamma x_2x_3-l^2=0$, the definition of l being given in equation (71). This conic touches the asymptotic tangents through P_x at the points where the flex-ray of the pencil p_λ crosses them. LANE, 1922. 3, p. 293

12. At any point P_x on a surface, the ray of any conjugate net $N_{\lambda h}$ of a pencil p_λ intersects the associate ray of $N_{\lambda h}$ in a point Q on the flex-ray of the pencil p_λ. The flex-ray and the line joining P_x to the point Q separate the ray and associate ray harmonically.

13. The dualistic correspondence which converts each point P_x of a surface S into the tangent plane of S at P_x is represented analytically by the transformation

$$\xi_1=x_4\,, \qquad \xi_2=-x_3\,, \qquad \xi_3=-x_2\,, \qquad \xi_4=x_1+(\beta\gamma+\theta_{uv})x_4$$

accompanied by the substitution

$$\begin{pmatrix} \beta & \gamma & p & q & dv/du \\ -\beta & -\gamma & \pi & \chi & -dv/du \end{pmatrix}.$$

LANE, 1926. 3, p. 206

14. The locus of the axes, at a point P_x of a surface, of all the conjugate nets of a pencil p_λ is a quadric cone (called *the axis quadric cone* of the pencil p_λ at the point P_x). Its equations are $\xi_1=4\beta\gamma\xi_2\xi_3-m^2=0$, the definition of m being given in equation (75). The tangent planes of this cone along the asymptotic tangents through P_x intersect in the cusp-axis of the pencil p_λ.

15. If the coordinates ξ of the tangent plane at a point x of a surface are defined by $\xi=e^{-\theta}(x,\, x_u,\, x_v)$, and if x satisfies system (6), then the differential equations of the surface in plane coordinates are

$$\begin{aligned} \xi_{uu} &= \pi\xi+\theta_u\xi_u-\beta\xi_v\,, \\ \xi_{vv} &= \chi\xi-\gamma\xi_u+\theta_v\xi_v \qquad (\theta=\log\beta\gamma)\,. \end{aligned}$$

16. The quadric (86) containing the asymptotic tangents at a point P_x of a surface has third-order contact with the asymptotic curves at P_x in case $k_1=-3$. It has fourth-order contact with the curve C_u in case $k_1=-3$, $k_2=-\varphi/2$, and has fourth-order contact with both asymptotic curves C_u, C_v in case $k_1=-3$, $k_2=-\varphi/2$, $k_3=-\psi/2$. Determine the quadric (86) that has sixth-order contact with C_u.

17. Consider a point P_1 near a point P_x on a surface S, and through P_1 draw a line P_1R meeting the quadric (86) in O_1, O_2, and meeting in T the tangent plane π of S at P_x. Let O_1 be the intersection which approaches P_x with P_1. Then as P_1 approaches P_x the limit of the cross ratio (P_1O_1TR) is $-k_1$; whatever be the curve along which P_1 approaches P_x, provided that it is not tangent to an asymptotic at P_x; whatever be the line P_1R, provided that it approaches a limit not in π; and whatever be R, provided that it approaches a limit distinct from P_x.

BOMPIANI, 1924. 1, p. 417

18. (*The quadric of Moutard* at a point P_x of a surface S and in the direction of the tangent to the curve $dv-\lambda du=0$ through P_x is defined to be the locus of the osculating conics at P_x of the curves of intersection of S and the planes of a pencil with the tangent as axis.) Show by use of the series (18) that for the quadric of Moutard the coefficients in equation (86) have the values given by

$$k_1=-1\,, \qquad k_2=(\gamma\lambda^3-2\beta)/3\lambda\,, \qquad k_3=-(2\gamma\lambda^3-\beta)/3\lambda^2\,,$$

$$k_4=-[4\beta^2-3\beta\varphi\lambda+12\beta\psi\lambda^2+(8\beta\gamma+18\theta_{uv})\lambda^3+12\gamma\varphi\lambda^4-3\gamma\psi\lambda^5+4\gamma^2\lambda^6]/36\lambda^3\,.$$

Two of the triple-point tangents of the curve of intersection of the surface S and the quadric of Moutard coincide, and the double tangent cuts the curve of intersection in five coincident points at the point P_x. At a point P_x on a surface S there are six tangents whose quadrics of Moutard pass through any given point not in the tangent plane of S at P_x, and consequently there are at P_x six plane sections of S whose osculating conics at P_x pass through the given point. Through each tangent at P_x there are two plane sections of S each of which has a six-point conic at P_x. Through any line l_1 at P_x there are nine plane sections of S each of which has a six-point conic at P_x. At P_x there are ordinarily twenty-seven plane sections each of which has a seven-point conic at P_x. At any point P_x on a surface S the locus of the osculating conics of the plane curves of section of S and the planes through a fixed line l_1 is an algebraic surface of the eighth order. Discuss this surface.

MOUTARD, 1880. 3; oral communication to the Société Philomathique, 1865; letter to Poncelet, 1863. DARBOUX, 1880. 2, pp. 349 and 363–72. WILCZYNSKI, 1909. 2, pp. 279 and 288

19. The correspondence between a variable point M in the tangent plane at a point P_x of a surface and its polar plane with respect to the quadric of Moutard for the tangent conjugate to the line P_xM is Segre's transformation. The correspondence between M and its polar plane with respect to the quadric of Moutard for the tangent P_xM is the transformation of Čech for which $k=-1/3$.

20. Show by the series (18) that all the ∞^4 non-composite cubic surfaces having fourth-order contact with a surface S at a point P_x cut the tangent plane of S at P_x in the same nodal cubic curve,

$$x_4=(4x_1+\varphi x_2+\psi x_3)x_2x_3-4(\beta x_2^3+\gamma x_3^3)/3=0 \ .$$

The line of inflexions of this curve is the second canonical edge e_2 of Green.

B. SEGRE, 1927. 5, p. 729

21. If a curve on an unspecialized surface in ordinary space is both a union curve for a congruence Γ_1 and a dual union curve for the reciprocal congruence Γ_2, then the curve is a curve of Segre and the congruences are somewhat restricted. Moreover, there is a one-parameter family of Segre curves which have the designated property. If more than a single one-parameter family of curves have this property, then the congruences are the flex-ray and cusp-axis congruences of the Segre-Darboux pencil, and every Segre curve has the property.

22. If the union curves of a congruence Γ_1 contain a conjugate net, then Γ_1 is the axis congruence of the net. Determine all the conjugate nets in the union curves of a congruence Γ_1. GREEN, 1919. 1, p. 133; BOMPIANI, 1924. 4, p. 11

23. The three ray-points, corresponding to a point P_x, of the three curves of Segre that pass through P_x lie on the second axis a_2 of Čech.

24. At a point P_x of a surface the osculating planes of any two Darboux curves and of the Segre curve conjugate to the third Darboux curve intersect in a straight line. BOMPIANI, 1924. 2, p. 51

25. The ray-point cubic (70) and the ray conic (see Ex. 11) are tangent at three points on the tangents of Segre, the common tangents of the two curves being given by

$$l=\epsilon(\beta^2\gamma)^{1/3}x_2+\epsilon^2(\beta\gamma^2)^{1/3}x_3 \qquad (\epsilon^3=1) .$$

26. (*The axis curves* of a conjugate net are defined to be the Γ_1-curves of the axis congruence of the net, and *the ray curves* to be the Γ_2-curves of the ray congruence of the net.) Show that the axis curves of a conjugate net N_λ themselves form a conjugate net in case λ satisfies the equation

$$2(\log \lambda)_{uv}+(\beta/\lambda^2)_u-(\gamma\lambda^2)_v=0 ,$$

and show also that the ray curves form a conjugate net in case λ satisfies the same equation with the signs of β and γ changed. If both the axis curves and the ray curves form conjugate nets then $(\log \lambda)_{uv}=0$. (A conjugate net N_λ with such a λ is called *isothermally conjugate.*) For an isothermally conjugate net *the cusp-axis curves* (corresponding to the developables of the cusp-axis congruence of the pencil determined by the net) themselves form a conjugate net. By means of the transformation (11) obtain $\lambda=1$, so that the surface sustaining a conjugate net whose ray curves and axis curves both form conjugate nets is restricted by the condition $\beta_u=\gamma_v$.

27. The extremals of the invariant integral $\int(\beta/v')du$ are hypergeodesics for which $A=D=0$, $B=(\log \beta)_u/2$, $C=(\log \beta)_v$, and for which the cusp-axis is *a scroll directrix of Sullivan*, namely, the line l_1 for which $a=\psi/2$, $b=\varphi/4$. Similarly, the extremals of the invariant integral $\int\gamma v'^2du$ are hypergeodesics for which $A=D=0$, $B=-(\log \gamma)_u$, $C=-(\log \gamma)_v/2$, and for which the cusp-axis is *the other scroll directrix of Sullivan*, namely, the line l_1 for which $a=\psi/4$, $b=\varphi/2$.
SULLIVAN, 1915. 2, p. 202; FUBINI and ČECH, 1927. 1, p. 684

28. At a point P_x on a surface the plane determined by the two scroll directrices of Sullivan (see Ex. 27) intersects the canonical plane in the canonical line for which $k=-3/8$; this is the line of intersection of the osculating planes of the three pangeodesics tangent to the curves of Segre at P_x. The canonical line for which $k=-3/4$ joins P_x to the point of intersection, distinct from P_x, of the quadrics of Moutard (see Ex. 18) for the three tangents of Segre, while the canonical line for which $k=-5/12$ joins P_x to the residual point of intersection of the quadrics of Moutard for the three tangents of Darboux.

29. If $\varphi=\psi=0$, then $\beta=UV$, $\gamma=h/U^{1/2}V^2$, where h is an arbitrary constant and U is an arbitrary function of u alone, and V of v alone. By means of the transformation (11) obtain $\beta=\gamma=1$ and by means of the integrability conditions (10) obtain $q_u=p_v=0$, $q_v=p_u$, so that the equations (6) become

$$x_{uu}=(lu+m)x+x_v , \qquad x_{vv}=(lv+n)x+x_u \qquad (l,\ m,\ n=\text{consts.}) .$$

(An integral surface of these equations is called *a coincidence surface* because it has certain coincidence properties.) Show that at each of its points the canonical lines of the first kind coincide, and similarly those of the second kind also coincide, the canonical plane and canonical point being indeterminate. A surface is a coincidence surface if, and only if, on it the curves of Darboux are projective geodesics. The algebraic equation of a coincidence surface for which $l=0$ can be obtained by integrating the differential equations of the surface by means of elementary functions.

WILCZYNSKI, 1913. 2; BOMPIANI, 1924. 2, p. 53

30. Differentiating equation (51) with respect to u by means of formulas (106), prove that two consecutive osculating linear complexes at a point P_x of a curve C_u ordinarily intersect in the linear congruence $\omega_{23}-\omega_{14}=\omega_{34}=0$, which consists of the tangents of the surface R_u (see Ex. 6) at the points of the u-tangent through the point P_x. The curve C_u belongs to a linear complex in case $\beta\gamma-(\log \beta)_{uv}=0$. Similarly, C_v belongs to a linear complex in case $\beta\gamma-(\log \gamma)_{uv}=0$. If both C_u and C_v belong to linear complexes, then by the transformation (11) obtain $\beta=\gamma$, and by integrating the equation $\beta^2-(\log \beta)_{uv}=0$ obtain $\beta=(U'V')^{1/2}/(U+V)$. Find p, q from the integrability conditions (10).

31. Prove that any net N_λ of curves on a surface S, such that at each point P_x of S the two curves of N_λ form with the parametric asymptotic curves a constant cross ratio r, is represented by the equation $(dv-\lambda du)(dv-r\lambda du)=0$. Generalize the theory of conjugate nets, calling the class of all nets N_λ, for every λ and a fixed r, a *bundle* B_r, and noting that the conjugate nets on S constitute the bundle B_{-1}. Define a *pencil* p_λ of nets in the bundle B_r so that any net $N_{\lambda h}$ of the pencil is represented by the equation $(dv-\lambda h du)(dv-r\lambda h du)=0$. Define the *ray-point cubic* of p_λ, and prove that it is independent of r. Prove that the envelope of the rays of all nets $N_{\lambda h}$ in the pencil p_λ is a curve of class four and order six, with a double point at P_x, the double-point tangents being the asymptotic tangents. This curve becomes the ray-conic, counted multiply, when $1+r=0$. When $1+r+r^2=0$, and only then, the curve is tangent to the flex-ray of the ray-point cubic. Then the curve has the flex-ray for triple tangent, the contact points being where the Segre tangents meet the flex-ray. This curve has no inflexions, four double points, and six cusps.

LANE, 1926. 4, p. 158

32. (If the directrix curves on a surface form a conjugate net the surface is called *isothermally-asymptotic*.) Show that by a transformation of parameters it is possible to make $\beta=\gamma$ in this case (see § 51 and Ex. 20 of Chap. V).

33. If a curve is such that its asymptotic osculating quadrics at each of its points intersect only in the asymptotic tangents at the point, the curve is a curve of Darboux.

34. When the parametric curves $dudv=0$ on a surface S are any whatever, provided that they do not form a conjugate net, any line l_2 in the tangent plane of S at a point P_x can be defined as joining the points x_u-bx, x_v-ax. The tangent plane at the point x_u-bx to the ruled surface of u-tangents constructed at the points of the

v-curve through P_x is determined by the points x, x_u, $x_{uv}-bx_v$. Similarly, the tangent plane at the point x_v-ax to the ruled surface of v-tangents at the points of the u-curve through P_x is determined by x, x_v, $x_{uv}-ax_u$. The line l_1 of intersection of these planes joins the points x and $-ax_u-bx_v+x_{uv}$. The two lines l_1, l_2, which are in *the relation* R, are in the reciprocal polar relation with respect to the quadric of Lie at the point P_x of the surface S if the parametric curves are the asymptotic curves on S.

GREEN, 1919. 1, p. 86; 1916. 3, p. 274

35. (A surface is called *a projectively minimal surface* in case it is an extremal of the invariant integral $\int\int \beta\gamma\, dudv$.) Such a surface is characterized by the property that on it and the four other sheets of the envelope of its quadrics of Lie the asymptotic curves correspond. Another characteristic property of a projectively minimal surface is as follows: the four edges of the tetrahedron of Demoulin that lie on the quadric of Lie at a point of a surface S generate congruences whose developables correspond to conjugate nets on S if, and only if, S is a projectively minimal surface.

THOMSEN, 1928. 4; 1921. 1, p. 233 and § 6

36. At a point P_x of a surface S the quadric of Darboux

$$2(x_2x_3-x_1x_4)-[\beta\gamma(1-k)+\theta_{uv}]x_4^2=0 \qquad (k=\text{const.})$$

is the quadric of Lie if $k=0$, the canonical quadric of Wilczynski if $k=1$, and a quadric of Fubini (see Ex. 8) if $k=1/3$. If any line l_1 meets this quadric, besides at P_x also at the point P_k, if l_1 meets the quadric of Lie at the point P_0, and meets the quadric of Wilczynski at the point P_1, then

$$(P_xP_0P_1P_k)=k\ ,$$

so that by means of this cross ratio the quadric of Darboux is characterized geometrically for an arbitrary value of k. Use equations (102) to study the envelope of the quadric of Darboux with an arbitrary value of k as the point P_x varies over the surface S.

LANE, 1926. 3, p. 208; BOMPIANI, 1927. 11, p. 189

37. The osculating plane (33) at a point P_x of a curve C_λ of the family $dv-\lambda du=0$ on a surface intersects the quadric (29) in a conic. The locus of this conic, as the curve C_λ varies but remains tangent to a fixed line t at P_x is the quadric

$$2\lambda^3(x_2x_3-x_1x_4)+4\beta\lambda x_4(x_3-\lambda x_2)+[\beta(-2\beta+\varphi\lambda-2\psi\lambda^2)-\theta_{uv}\lambda^3]x_4^2=0\ .$$

When $\lambda\to\infty$ this quadric approaches the canonical quadric of Wilczynski. When the quadric (31) is used in place of (29), the corresponding locus is the quadric

$$2(x_2x_3-x_1x_4)-4\gamma\lambda x_4(x_3-\lambda x_2)+[\gamma(-2\gamma\lambda^3+\psi\lambda^2-2\varphi\lambda)-\theta_{uv}]x_4^2=0\ ,$$

which is the quadric of Wilczynski when $\lambda=0$.

BOMPIANI, 1929. 2, p. 680

38. The two quadrics in Exercise 37 intersect, besides in the asymptotic tangents at the point P_x, also in a conic which lies in the plane

$$4\lambda(\beta+\gamma\lambda^3)(x_3-\lambda x_2)+[2(\gamma^2\lambda^6-\beta^2)+\varphi\lambda(\beta+2\gamma\lambda^3)-\varphi\lambda^2(2\beta+\gamma\lambda^3)]x_4=0 \ .$$

When the tangent t varies through P_x, this plane envelops the cone (99) of Segre.

39. Show that the constant c in the transformation (11) can be chosen so that the determinant (x, x_u, x_v, x_{uv}) has the value $e^{2\theta}$. Placing $X=x_1x+x_2x_u+x_3x_v+x_4x_{uv}$, prove that $x_1=(X, x_u, x_v, x_{uv})e^{-2\theta}$, $x_2=(x, X, x_v, x_{uv})e^{-2\theta}$, etc., and then deduce the formulas (102) by direct differentiation.

40. (A *Roman surface of Steiner* is a quartic surface each of whose tangent planes cuts it in two conics.) The parametric equations of the unique osculating Roman surface of Steiner, which has contact of the fourth order, at a point P_x of a surface S can be written in the form

$$\begin{aligned}
&x_1=24-3(2\beta\psi+\varphi^2/8)\xi^2+(3\varphi\psi/4-40\beta\gamma/3-12\theta_{uv})\xi\eta-3(2\gamma\varphi+\psi^2/8)\eta^2 \ ,\\
&x_2=24\xi+3\varphi\xi^2-3\psi\xi\eta+8\gamma\eta^2 \ ,\\
&x_3=24\eta+8\beta\xi^2-3\varphi\xi\eta+8\psi\eta^2 \ ,\\
&x_4=24\xi\eta \ .
\end{aligned}$$

DARBOUX, 1880. 2, p. 359

CHAPTER IV

CONJUGATE NETS

Introduction. The geometric theory of conjugate nets began with Dupin, but the analytic theory which commonly receives a geometrical interpretation in connection with parametric conjugate nets is older and goes back to Laplace and Euler. If a complete list were made of all the mathematicians who have enriched the theory of conjugate nets by noteworthy contributions, the list would include nearly, if not quite, all of the most distinguished geometers of the latter part of the nineteenth and the early part of the twentieth centuries. Some of these would be Koenigs, L. Lévy, Goursat, Darboux, Guichard, Ribaucour, Demoulin, Tzitzéica, Bianchi, Segre, Bompiani, Wilczynski, Green, and Eisenhart.

For the purposes of this and the following chapters it is expedient to revise our definition of a conjugate net. In Section 19 a net of curves on a surface in ordinary space was said to be a conjugate net in case at each point of the surface the two tangents of the curves of the net separate harmonically the asymptotic tangents of the surface, and it was proved in Section 21 that the tangents of the curves of one family of a conjugate net constructed at the points of each fixed curve of the other family form a developable surface. But in space of more dimensions than three, a nondevelopable surface ordinarily has on it no asymptotic curve, and even if asymptotic curves exist on such a surface they do not form a net. Therefore we shall take the second property of a conjugate net just mentioned as our definition of a conjugate net in hyperspace. In ordinary space the two definitions are equivalent, but in hyperspace the definition that we are now adopting is more fruitful.

Whenever there is no danger of ambiguity a *conjugate net* may be called for brevity simply a *net*. In this sense *a net* is what is called in French *un réseau*. The net under consideration will be taken as parametric throughout most of this chapter.

The contents of this chapter are disclosed by the following remarks. In presenting the theory of parametric conjugate nets it is natural, first of all, to introduce *the equation of Laplace* which is associated with a parametric conjugate net, and then to make inquiries as to what surfaces are capable of sustaining conjugate nets. The analytic theory of *the transformation of Laplace*, as interpreted by Darboux, leads to a sequence of conjugate nets determined by a given net and called a *Laplace sequence*. One interesting

class of problems is concerned with the conditions under which such a sequence can terminate. These and related matters occupy Sections 26–28. The theory of nets in the plane is taken up in Section 29, and conjugate nets in ordinary space are investigated in Sections 30–31. The conjugate and harmonic relations of nets and congruences are studied in Section 32, and the chapter closes with a brief account of the polar relation of two Laplace sequences with respect to a hyperquadric.

26. A surface referred to a conjugate net. The definition of a conjugate net which we are now employing may be stated as follows. *A net of curves* (as defined in Section 8) *on a surface in a linear space of n dimensions is a conjugate net in case the tangents of the curves of one family of the net constructed at the points of each fixed curve of the other family form a developable surface.* Although the two families do not enter this definition symmetrically, it will soon be shown that a conjugate net is symmetric in its two families. The purpose of this section is to introduce *the Laplace equation* that is associated with a parametric conjugate net, and to deduce a few of the simplest properties of this equation. In particular, *the Laplace-Darboux invariants H, K* will be defined.

Let us consider a surface S with the parametric vector equation $x=x(u, v)$ in space S_n. We shall now prove that *a necessary and sufficient condition that the parametric curves on the surface S form a conjugate net is that x satisfies an equation of Laplace.*

$$x_{uv}=cx+ax_u+bx_v \,, \tag{1}$$

where a, b, c are scalar functions of u, v. Let us first suppose that the parametric curves on S form a conjugate net. The point P_y defined by

$$y=x_v+kx \qquad (k \text{ scalar})$$

is on the v-tangent at the point P_x. As u varies, the point P_y generates a curve whose tangent at P_y is determined by P_y and the point y_u given by

$$y_u=x_{uv}+kx_u+k_ux \;.$$

This point is on the v-tangent in case y_u is a linear combination of x and x_v, say $fx+gx_v$. Therefore, if the parametric curves form a conjugate net, then x satisfies the equation

$$x_{uv}+kx_u+k_ux=fx+gx_v \,,$$

which is certainly an equation of Laplace. Conversely, if x satisfies equation (1) and if P_y is defined as above, then we find

$$y_u = (a+k)x_u + bx_v + (c+k_u)x \ .$$

The point y_u is on the v-tangent if we take $k=-a$. Therefore this tangent generates a developable surface when u varies. A similar argument can be made with the u-curves and v-curves interchanged. This completes the proof. The parametric conjugate net on the surface S can consistently be denoted by N_x, since S is generated by a point P_x. The symmetry of equation (1) in u and v shows that the two families really play symmetrical rôles in a conjugate net.

From the foregoing demonstration it follows that the points x_1 and x_{-1} defined by

$$x_1 = x_v - ax \ , \qquad x_{-1} = x_u - bx \tag{2}$$

are what would be called in the language of Section 21 *the ray-points* of the u-curve and the v-curve respectively, corresponding to the point P_x. These points are also called respectively *the first and minus-first Laplace transforms (or transformed points) of the point P_x with respect to the net N_x.* The Laplace transformation defined by equations (2) will be discussed more at length in Section 28.

The effect of the transformation of proportionality factor,

$$x = \lambda \bar{x} \qquad (\lambda \text{ scalar} \neq 0) \ , \tag{3}$$

on equation (1) is to produce another equation of the same form whose coefficients, indicated by dashes, can easily be shown by direct calculation to be given by

$$\begin{cases} \bar{a} = a - (\log \lambda)_v \ , \qquad \bar{b} = b - (\log \lambda)_u \ , \\ \lambda \bar{c} = -\lambda_{uv} + a\lambda_u + b\lambda_v + c\lambda \ . \end{cases} \tag{4}$$

This transformation leaves the net N_x unchanged. It will be observed that if λ is a solution of equation (1), then $\bar{c}=0$. Moreover, the net N_x is also left unchanged by the transformation of parameters

$$\bar{u} = U(u) \ , \qquad \bar{v} = V(v) \qquad (U'V' \neq 0) \ , \tag{5}$$

which converts equation (1) into another equation of the same form, whose coefficients, again indicated by dashes, are given by

$$\bar{a}=a/V' , \qquad \bar{b}=b/U' , \qquad \bar{c}=c/U'V' . \tag{6}$$

The Laplace-Darboux invariants H, K of the net N_x are defined* by the formulas

$$H=c+ab-a_u , \qquad K=c+ab-b_v . \tag{7}$$

These functions can easily be shown to be absolute invariants under the transformation (3), and are relative invariants under (5), being transformed by (5) according to the formulas

$$\bar{H}=H/U'V' , \qquad \bar{K}=K/U'V' . \tag{8}$$

The invariants H, K are sometimes spoken of simply as *the invariants* of the net.

If $H=K$ then $b_v=a_u$ and there exists a function λ such that

$$(\log \lambda)_v=a , \qquad (\log \lambda)_u=b .$$

If this function λ is used in the transformation (3), then $\bar{a}=\bar{b}=0$, as is seen from equations (4). Therefore *a Laplace equation with equal invariants can be reduced to the form*

$$x_{uv}=cx . \tag{9}$$

27. Surfaces sustaining conjugate nets. There exist surfaces in hyperspace that do not have conjugate nets on them. The aim of this section is to show under just what conditions a surface may sustain one or more conjugate nets.

In the definition stated in Section 10 for the space $S(k, 0)$ at a point of a ruled surface the ruled surface can be replaced by any surface immersed in space S_n. In particular, *the space* $S(2, 0)$ *at a point* P_x *of such a surface* S *is the ambient of the osculating plane at* P_x *of every curve on* S *through* P_x. Since the osculating plane of a curve at the point P_x is determined by the points x, x', x'', we see by letting λ, λ' vary in equations (III, 32) that *the space* $S(2, 0)$ *is ordinarily the space* S_5 *determined by the six points* x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}. However, *in case* x *satisfies a differential equation of the form*

$$Ax_{uu}+2Bx_{uv}+Cx_{vv}+Dx_u+Ex_v+Fx=0 , \tag{10}$$

* Darboux, 1889. 1, Chap. II, p. 23.

in which the six coefficients are scalar functions of u, v and not all of A, B, C are zero, the space $S(2, 0)$ is at most a space S_4.

We shall now show that *every integral surface of an equation of the form* (10) *sustains a conjugate net or else a one-parameter family of asymptotic curves.* For this purpose let us consider in space S_n a surface generated by a point P_x with x satisfying equation (10) and let us carry out the transformation of parameters

$$\xi=\xi(u, v), \qquad \eta=\eta(u, v) \qquad (J=\xi_u\eta_v-\xi_v\eta_u\neq 0). \tag{11}$$

Differentiating x by the rules of elementary calculus we obtain

$$\begin{cases} x_u=x_\xi\xi_u+x_\eta\eta_u, \qquad x_v=x_\xi\xi_v+x_\eta\eta_v, \\ x_{uu}=x_{\xi\xi}\xi_u^2+2x_{\xi\eta}\xi_u\eta_u+x_{\eta\eta}\eta_u^2+x_\xi\xi_{uu}+x_\eta\eta_{uu}, \\ x_{vv}=x_{\xi\xi}\xi_v^2+2x_{\xi\eta}\xi_v\eta_v+x_{\eta\eta}\eta_v^2+x_\xi\xi_{vv}+x_\eta\eta_{vv}, \\ x_{uv}=x_{\xi\xi}\xi_u\xi_v+x_{\xi\eta}(\xi_u\eta_v+\xi_v\eta_u)+x_{\eta\eta}\eta_u\eta_v+x_\xi\xi_{uv}+x_\eta\eta_{uv}. \end{cases} \tag{12}$$

Substituting in equation (1) these expressions for the derivatives of x we get another equation of the same form as (10), the first three of whose coefficients, indicated by dashes, are given by

$$\begin{cases} \bar{A}=A\xi_u^2+2B\xi_u\xi_v+C\xi_v^2, \\ \bar{B}=A\xi_u\eta_u+B(\xi_u\eta_v+\xi_v\eta_u)+C\xi_v\eta_v, \\ \bar{C}=A\eta_u^2+2B\eta_u\eta_v+C\eta_v^2. \end{cases} \tag{13}$$

Incidentally it is easy to verify that

$$\overline{AC}-\bar{B}^2=J^2(AC-B^2), \tag{14}$$

and also that the determinant of the coefficients of A, B, C in the right members of equations (13) is J^3. Consequently not all of $\bar{A}$, $\bar{B}$, $\bar{C}$ are zero.

There are two cases to be discussed, according as $AC-B^2\neq 0$ or $AC-B^2=0$. If $AC-B^2\neq 0$, the equation

$$A\theta_u^2+2B\theta_u\theta_v+C\theta_v^2=0 \tag{15}$$

has two functionally independent solutions,

$$\theta_1=\theta_1(u, v), \qquad \theta_2=\theta_2(u, v).$$

If these solutions are used as the functions ξ, η in the transformation (11), then equations (13) show that $\bar{A}=\bar{C}=0$. Since $\bar{B}\neq 0$, equation (10) is thus

transformed into an equation of Laplace with independent variables ξ, η. It follows that *the ξ-curves and η-curves form a conjugate net.* The parametric conjugate net $d\xi d\eta=0$ associated with this equation of Laplace is the same net that in the original parameters u, v has the curvilinear differential equation

$$Cdu^2-2Bdudv+Adv^2=0\ , \tag{16}$$

as can be verified by calculating the product $d\xi d\eta$ from equations (11) and then making use of the well-known properties of the sum and product of the roots of the quadratic equation (15).

If $AC-B^2=0$, equation (15) has only one independent solution, $\theta_1=\theta_1(u, v)$. With this function θ_1 in place of ξ and with $\eta=v$ in the transformation (11) we obtain $\bar{A}=0$, and therefore $\bar{B}=0$, $\bar{C}\neq 0$. It follows that x satisfies an equation of the form

$$x_{\eta\eta}+\bar{D}x_\xi+\bar{E}x_\eta+\bar{F}x=0\ . \tag{17}$$

Consequently *the η-curves are asymptotic curves,* since the osculating plane at every point of each η-curve coincides with the tangent plane of the surface at the point. Thus the demonstration of the theorem is completed.

Conversely, if a surface S immersed in space S_n has on it a conjugate net, or else a one-parameter family of asymptotics, and if the parameters on S are suitably chosen, then x satisfies an equation of the form (1), or else of the form (17), each of which is of the form (10). Then the space $S(2, 0)$ is at most a space S_4. We reach thus the following result:

A surface with the parametric vector equation $x=x(u, v)$ immersed in space S_n sustains a conjugate net or else a one-parameter family of asymptotic curves if, and only if, x satisfies an equation of the form (10); *in geometrical language, if, and only if, the space $S(2, 0)$ at each point of the surface is at most a space S_4.*

Contemplation of equation (10) leads to the following generalities. *An unspecialized surface immersed in a space S_n with $n>4$ does not sustain a conjugate net,* since the six or more homogeneous coordinates of a variable point on the surface cannot ordinarily be made to satisfy an equation of the form (10), whose coefficients yield only five essential ratios to be disposed of. However, the class of surfaces each of which does sustain a conjugate net in such a space is of considerable interest. *On a non-developable surface immersed in space S_4 there is a unique conjugate net, or else a unique one-parameter family of asymptotics,* since the five homogeneous coordinates of a variable point on the surface can be made to satisfy a unique equation (see Exs. 2, 3, 4) of the form (10). *On a surface immersed in ordinary space S_3*

there are infinitely many conjugate nets, since the four homogeneous coordinates of a variable point on the surface can be made to satisfy an equation of the form (10) with one coefficient arbitrary. In fact, one of the families of a conjugate net can be assigned arbitrarily on such a surface; if the surface is not developable there are also two distinct one-parameter families of asymptotic curves on it; if the surface is developable, the generators are the only asymptotic curves that the surface sustains.

28. The transformation of Laplace. Equation (1) ordinarily admits of two transformations into other equations of the same kind. These transformations are defined by equations (2), and are distinguished as being in the positive or in the negative direction. Each of the new equations ordinarily admits of a similar transformation in one direction back into the original equation and in the other direction into still another equation of the same kind, and so on. These transformations are called *the transformations of Laplace* because Laplace used them in investigating the possibility of integrating equation (1) by quadratures. The present section is devoted to a study of these transformations, which are referred to generically as *the transformation of Laplace.*

More specifically, the contents of this section may be summarized as follows. The first problem is to deduce formulas for the r'th and $-r$'th Laplace transforms of equation (1). This analytic theory received at the hands of Darboux a geometrical interpretation in the theory of the so-called *Laplace sequence* of conjugate nets determined by a given net. After introducing the Laplace sequence, some geometrical relations of the various nets of a sequence are studied, and the possibilities under which a Laplace sequence can terminate are explained.

The analytic theory of the Laplace transformation will now be briefly studied. Let us consider in space S_n a surface S, with the parametric vector equation $x=x(u, v)$, referred to a conjugate net N_x so that x satisfies equation (1). On the surface S let us consider a general point P_x and the parametric curve C_u through P_x. The ray-point x_1 of C_u corresponding to P_x (or first Laplace transformed point of P_x with respect to the net N_x) is defined by the first of equations (2). Differentiating x_1 with respect to u and using equations (1), (7) we find

$$x_{1u}=bx_1+Hx\ . \tag{18}$$

If $H=0$, the point x_1 is fixed when u varies, so that the locus of this point as u, v vary is a v-curve. If $H\neq 0$, the locus of the point x_1 is a surface on which the parametric curves form a conjugate net N_1, since x_1 satisfies a

Laplace equation, whose coefficients, invariants, and transforms, indicated by the appropriate subscripts, are given by

$$(19)\qquad \left\{\begin{array}{ll} a_1=a+(\log H)_v\,, & b_1=b\,,\\ c_1=c+H-K-b\,(\log H)_v\,, & \\ H_1=2H-K-(\log H)_{uv}\,, & K_1=H\,,\\ x_2=(x_1)_1=x_{1\,v}-a_1x_1\,, & \\ (x_1)_{-1}=x_{1\,u}-b_1x_1=Hx\,. & \end{array}\right.$$

The last of these equations shows that *just as the point* x_1 *is the ray-point of the curve* C_u *of the net* N_x, *so the point* x *is the ray-point of the curve* C_v *of the net* N_1.

It is customary to define *a congruence of lines in hyperspace to be* ∞^2 *lines which can be assembled into* ∞^1 *developable surfaces in two ways.* Just as in ordinary space, the lines of such a congruence are the common tangents of two surfaces called *focal surfaces* of the congruence. The developables of the congruence determine on the focal surfaces two conjugate nets called the *focal nets* of the congruence. With these ideas in mind we see that *the nets* N_x *and* N_1 *are the focal nets of the congruence of lines* xx_1 *which are the* v*-tangents of* N_x *and are at the same time the* u*-tangents of* N_1.

The foregoing considerations for the net N_x can be repeated for the net N_1 when $H\neq 0$. The equation for the net N_1 analogous to (18) for the net N_x is

$$(20)\qquad x_{2\,u}=b_1x_2+H_1x_1\,.$$

If $H_1=0$, the locus of the point x_2 is a v-curve. If $H_1\neq 0$, the locus of this point is a net N_2, since x_2 satisfies a Laplace equation whose coefficients, invariants, and transforms, indicated by the appropriate subscripts, are given by

$$(21)\qquad \left\{\begin{array}{ll} a_2=a_1+(\log H_1)_v\,, \quad b_2=b_1=b\,, & \\ c_2=c_1+H_1-K_1-b_1(\log H_1)_v\,, & \\ H_2=2H_1-K_1-(\log H_1)_{uv}=3H-2K-(\log H^2H_1)_{uv}\,, & K_2=H_1\,,\\ x_3=(x_2)_1=x_{2\,v}-a_2x_2\,, & \\ (x_2)_{-1}=x_{2\,u}-b_2x_2=H_1x_1\,. & \end{array}\right.$$

And so, by continuation of the argument, if $HH_1\cdots H_{r-2}\neq 0$, we find

$$(22)\qquad x_r=x_{r-1\,v}-a_{r-1}x_{r-1}\,,\qquad x_{r\,u}=b_{r-1}x_r+H_{r-1}x_{r-1}\,,$$

and if $H_{r-1}\neq 0$ we obtain for x_r a Laplace equation whose coefficients and invariants are given by

$$(23)\qquad \begin{cases} a_r=a_{r-1}+(\log H_{r-1})_v\ , \qquad b_r=b_{r-1}=b\ , \\ c_r=c_{r-1}+H_{r-1}-K_{r-1}-b_{r-1}\,(\log H_{r-1})_v\ , \\ H_r=2H_{r-1}-K_{r-1}-(\log H_{r-1})_{uv} \\ \quad =(r+1)H-rK-(\log H^rH_1^{r-1}\cdots H_{r-2}^2H_{r-1})_{uv}\ , \\ K_r=H_{r-1} \qquad\qquad (r\geqq 1)\ . \end{cases}$$

Similarly, interchanging u and v, a and b, H and K, and changing the signs of the subscripts, if $KK_{-1}\cdots K_{-r+2}\neq 0$, we find

$$(24)\qquad x_{-r}=x_{-r+1\,u}-b_{-r+1}x_{-r+1}\ , \qquad x_{-r\,v}=a_{-r+1}x_{-r}+K_{-r+1}x_{-r+1}\ ,$$

and, if $K_{-r+1}\neq 0$, we obtain for x_{-r} a Laplace equation whose coefficients and invariants are given by

$$(25)\qquad \begin{cases} a_{-r}=a_{-r+1}=a\ , \qquad b_{-r}=b_{-r+1}+(\log K_{-r+1})_u\ , \\ c_{-r}=c_{-r+1}+K_{-r+1}-H_{-r+1}-a_{-r+1}(\log K_{-r+1})_u\ , \\ K_{-r}=2K_{-r+1}-H_{-r+1}-(\log K_{-r+1})_{uv} \\ \quad =(r+1)K-rH-(\log K^rK_{-1}^{r-1}\cdots K_{-r+2}^2K_{-r+1})_{uv}\ , \\ H_{-r}=K_{-r+1} \qquad\qquad (r\geqq 1)\ . \end{cases}$$

Let us observe more closely the geometrical aspects of the Laplace transformation. The sequence of nets

$$\ldots,\ N_{-r},\ \ldots,\ N_{-2},\ N_{-1},\ N_x,\ N_1,\ N_2,\ \ldots,\ N_r,\ \ldots,$$

each of which can be derived from the two adjacent nets by transformations of Laplace, is called a *sequence of Laplace* and is indicated schematically in Figure 19. If $H_r=0$, the sequence terminates in the positive direction, since N_{r+1} reduces to a v-curve, which is the locus of the vertices of cones circumscribing the net N_r along the u-curves thereon. Similarly, if $K_{-r}=0$, the sequence terminates in the negative direction, since N_{-r-1} reduces to a u-curve, which is the locus of the vertices of cones circumscribing the net N_{-r} along the v-curves thereon. It is natural to ask whether it is possible for the sequence to terminate under any other conditions than those just stated, which were first studied by Laplace* analytically. In order to

* Laplace, 1893. 1, p. 5. *Recherches sur le calcul intégral aux différences partielles,* "Mémoires de l'Académie Royale des Sciences de Paris" (1773).

answer this question let us suppose for simplicity that the locus N_1 reduces to a curve; then x_1 satisfies an equation of the form

$$Ax_{1u}+Bx_{1v}+Cx_1=0$$

with not all of the coefficients A, B, C equal to zero. This equation is equivalent to

$$Bx_{vv}+(bA-aB+C)x_v+[(H-ab)A-a_vB-aC]x=0\ .$$

If $B\neq 0$, then the surface S is ruled with its v-curves for generators. In fact, S is developable and N_1 reduces to a u-curve to which the generators

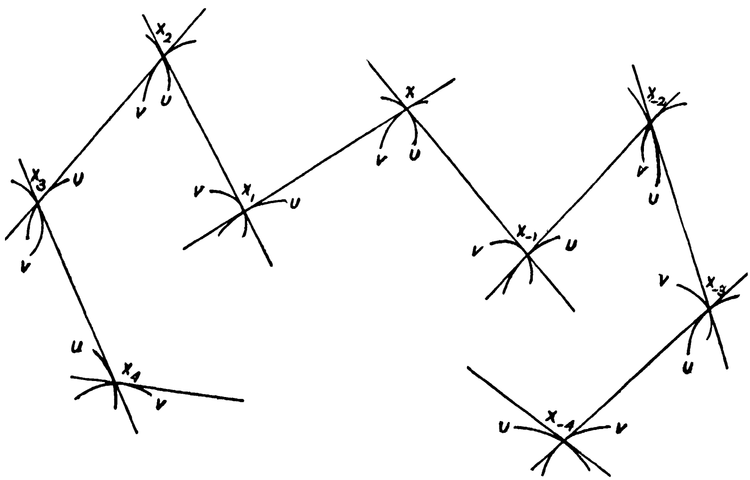

FIG. 19

of S are tangent (see Ex. 2). This case was first studied by Goursat.* If $B=0$, since the surface S is non-degenerate it follows that

$$bA+C=0\ , \qquad HA=0\ .$$

Since not both of A and C are zero it follows that $H=0$. This is the case of Laplace. It can happen that $B\neq 0$, $H=0$. In this so-called mixed† case, the locus N_1 reduces to a fixed point and the surface S is a cone with its vertex at this point, the generators of the cone being v-curves. We conclude

* Goursat, 1896. 1.

† Bompiani, 1912. 1, p. 393.

that *a sequence may terminate according to the case of Laplace, or according to the case of Goursat, or according to the mixed case.*

There are some interesting geometrical relations among the linear osculants constructed at corresponding points of curves of nets that belong to a sequence of Laplace. Equations (22), (24) show that *the tangent line at a point of a v-curve of any net in a Laplace sequence is also the tangent line at the corresponding point of the u-curve of the adjacent net in the positive direction of the sequence.* It is not difficult to show that *the osculating plane at a point of a v-curve of any net of a sequence is the tangent plane at the corresponding point of the surface sustaining the first transformed net, and is also the osculating plane at the corresponding point of the u-curve of the second transformed net.* For this purpose it is sufficient to demonstrate by simple calculations the truth of the equations

$$(x_1,\, x_{1\,u},\, x_{1\,uu})=H^2(x,\, x_u,\, x_v)\ , \qquad (x_{-1},\, x_{-1\,v},\, x_{-1\,vv})=-K^2(x,\, x_u,\, x_v)\ ,$$

in which parentheses denote determinants. This amounts to proving that every linear combination of x_1, $x_{1\,u}$, $x_{1\,uu}$ can be expressed as a linear combination of x, x_u, x_v; and that the same is true of every linear combination of x_{-1}, $x_{-1\,v}$, $x_{-1\,vv}$. In a similar way it can be shown that the osculating space S_3 at a point of a v-curve of a net is also the osculating space S_3 at the corresponding point of the u-curve of the third transformed net, being at the same time the space S_3 determined by the tangent planes of the first and second transforms. Moreover, this space S_3 may be regarded as determined by the tangent plane and the osculating plane of the v-curve of the first transformed net or else by the tangent plane and the osculating plane of the u-curve of the second transformed net. In general, *the osculating space S_k at a point of a v-curve of a net is the osculating space S_k at the corresponding point of the u-curve of the* kth *transformed net* ($k>0$). This space S_k may be regarded as determined in various ways; in particular, it can be regarded as determined by the tangent planes of the $k-1$ intervening transformed nets.

We shall next discuss some geometrical properties of a sequence that terminates according to the case of Goursat, proving first that *if the v-curves of a net in space S_n are hyperplane curves, the sequence terminates in the positive direction according to the case of Goursat after $n-1$ transformations.* If the v-curves of a net N_x in space S_n are hyperplane curves, then each v-curve lies in a space S_{n-1} and therefore the coordinates x satisfy an equation of the form

$$\sum_{i=1}^{n+1} U_i(u)x^{(i)}=0\ ,$$

which can be abbreviated into $\Sigma Ux=0$. Since $\Sigma Ux_v=0$ it follows that $\Sigma Ux_1=0$. Moreover, since

$$\Sigma U'x_1+\Sigma Ux_{1u}=0 \ ,$$

the accent indicating differentiation with respect to u, and since by equation (18) we have $\Sigma Ux_{1u}=0$, it follows that $\Sigma U'x_1=0$. Therefore each v-curve of the net N_1 lies in a space S_{n-2}, because x_1 satisfies the two equations

$$\Sigma Ux_1=0 \ , \qquad \Sigma U'x_1=0 \ .$$

Similarly, it can be shown that x_2 satisfies the three equations

$$\Sigma Ux_2=0 \ , \qquad \Sigma U'x_2=0 \ , \qquad \Sigma U''x_2=0 \ ,$$

so that each v-curve of the net N_2 lies in a space S_{n-3}. Continuation of the reasoning for a sufficient number of steps leads to the conclusion that the v-curves of the net N_{n-3} are plane curves, that the v-curves of the net N_{n-2} are straight lines, and then that the v-curves of the locus N_{n-1} reduce to points, so that the net N_{n-2} lies on a developable surface with a u-curve N_{n-1} for its edge of regression, and the sequence terminates according to the case of Goursat. The hyperplanes containing the v-curves of the net N_x are the osculating hyperplanes of the u-curve N_{n-1}. Conversely, if the sequence terminates in the positive direction according to the case of Goursat, with the locus N_{n-1} reducing to a u-curve, the osculating hyperplanes of this curve contain the v-curves of the net N_x.

The geometrical situation when a sequence in space S_n terminates according to the case of Laplace can be described briefly as follows. If the locus N_{n-1} reduces to a v-curve, so that the Laplace sequence terminates in the positive direction according to the case of Laplace after $n-1$ transformations, then $H_{n-2}=0$ and the second of equations (22) with $r=n-1$ gives

$$x_{n-1\,u}=bx_{n-1} \ .$$

By repeated use of the first of equations (22) with $r=n-1, \ldots, 1$ we find that x_{n-1} can be expressed as a linear combination of x_{n-2} and its first v-derivative, or as a linear combination of x_{n-3} and its first two v-derivatives, and so on until x_{n-1} is expressed as a linear combination of x and its first $n-1$ v-derivatives. Therefore, if the locus N_{n-1} reduces to a v-curve, then each point x_{n-1} of this curve is the vertex of a cone circumscribing the net N_{n-2} along a u-curve; the osculating planes of the v-curves at the points of the corresponding u-curve of the net N_{n-3} pass through the same point

x_{n-1}, as do also the osculating spaces S_3 of the v-curves at the points of the corresponding u-curve of the net N_{n-4}, and so on until finally the osculating hyperplanes of the v-curves at the points of the corresponding u-curve of the net N_x pass through the same point x_{n-1}. Conversely,* *if the osculating hyperplanes of the v-curves at the points of each u-curve of a net N_x pass through a point, then the sequence terminates in the positive direction according to the case of Laplace after $n-1$ transformations.*

29. Plane nets. After setting up the completely integrable system of partial differential equations that define a plane net except for a projective transformation, and writing the integrability conditions therefor, we shall limit the discussion in this section to two characterizations of plane nets with equal invariants, one of which is due to Koenigs and one to Green.

The form of the differential equations defining a plane net can be discovered very quickly by the following geometrical considerations. Let the $n+1$ homogeneous coordinates x of a point on a plane in space S_n be given as analytic functions of two independent variables u, v, and suppose that the parametric curves form a net in the sense of Section 8 in the portion of the plane under consideration. Every such net in a plane is a conjugate net according to the definition of conjugate net given in Section 26, and therefore x satisfies an equation of the form (1). Moreover, every curve in the plane is an asymptotic curve, according to the definition of asymptotic curve given in Section 8, so that x also satisfies a system of equations of the form (III, 1). Therefore *the $n+1$ coordinates x of a variable point in a plane referred to any net whatever and situated in space S_n are solutions of a system of equations of the form*

$$\left\{\begin{aligned} x_{uu} &= px+\alpha x_u+\beta x_v \ , \\ x_{uv} &= cx+ax_u+bx_v \ , \\ x_{vv} &= qx+\gamma x_u+\delta x_v \ . \end{aligned}\right. \tag{26}$$

It may be well to remark that, since the point x is restricted to lie in a plane, at most three of the coordinates x are linearly independent, and for some purposes the remaining coordinates can be disregarded. We shall disregard them from now on in this section. This amounts to placing $n=2$.

Of the four third derivatives of x there are two that can be calculated in two ways, since

$$(x_{uu})_v=(x_{uv})_u \ , \qquad (x_{uv})_v=(x_{vv})_u \ .$$

* Tzitzéica, 1924. 3, p. 129; Bompiani, 1912. 1, p. 389.

Each of these equations reduces by means of system (26) to a linear first-order equation of the form (II, 2) all of whose coefficients must be zero. Thus we obtain *six integrability conditions on the coefficients of system* (26):

$$
(27) \qquad \begin{cases}
p_v + q\beta + c\alpha = c_u + ap + bc\,, \\
q_u + p\gamma + c\delta = c_v + bq + ac\,, \\
\alpha_v + \beta\gamma = a_u + c + ab, \\
\delta_u + \beta\gamma = b_v + c + ab\,, \\
\beta_v + p + b\alpha + \beta\delta = b_u + a\beta + b^2\,, \\
\gamma_u + q + a\delta + \gamma\alpha = a_v + b\gamma + a^2\,.
\end{cases}
$$

Not only the original parametric net, but also any net into which it can be projected by a projective transformation of the plane, is an integral net of system (26). Conversely, the theory of linear partial differential equations teaches us that this system of equations with the integrability conditions (27) satisfied defines a plane net except for such a projective transformation.

We shall now prove a well-known theorem of Koenigs.* *The projection of the asymptotic net on a surface in ordinary space from a point onto a plane not containing the point is a plane net with equal invariants.* For this purpose let us consider in space S_3 a surface whose parametric vector equation is $x = x(u, v)$ referred to its asymptotic net. The four coordinates x satisfy equations (III, 1) with $\delta_u - \alpha_v = 0$. Let us project the point P_x from the point (0, 0, 0, 1) onto the plane $x_4 = 0$; the coordinates of the projection may be taken as the first three of the coordinates x. These three coordinates satisfy not only equations (III, 1) but also a Laplace equation and therefore satisfy a system of equations of the form (26). From the third and fourth of the integrability conditions (27) and from the definitions (7) we obtain

$$
(28) \qquad \delta_u - \alpha_v = b_v - a_u = H - K\,.
$$

Therefore, for the plane net which is the perspective of the asymptotic net, we have $H = K$, as was to be proved. The reader may convince himself that there is no essential loss of generality in the choice we have made of the center and plane of projection.

Green's geometric characterization† of a plane net with equal invariants has the advantage of depending on nothing outside of the plane of the net. He shows that ordinarily a given plane net determines uniquely a covariant

* Koenigs, 1892. 1. † Green, 1918. 3.

net, called *the congruentially associated net* of the given net, with the property that, as a point varies along a curve of the associated net, the harmonic conjugate of the tangent of this curve with respect to the tangents of the

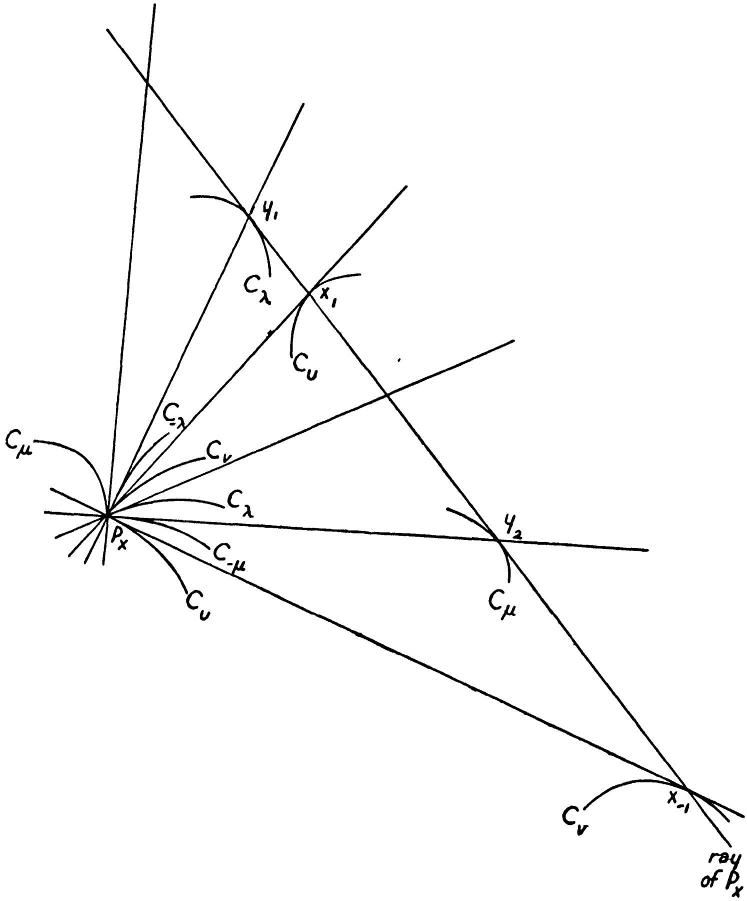

FIG. 20

given net at this point passes through the corresponding focal point of the ray of the given net. Before stating Green's characterization we shall find the curvilinear differential equation of the net congruentially associated with the parametric net of system (26). For this purpose let us consider a one-parameter family of curves $dv-\lambda du=0$ in the plane of Figure 20,

and the curve C_λ of this family through a piont P_x. The ray of P_x with respect to the parametric net joins the points x_1, x_{-1} defined by equations (2); consequently the point P_y defined by

$$y = x_{-1} + kx_1 \qquad (k \text{ scalar})$$

is on the ray. As the point P_x varies along the curve C_λ, the point P_y generates a curve whose tangent at P_y is determined by P_y and the point y' given by

$$y' = y_u + y_v\lambda = (P + K\lambda + Hk + Q\lambda k)x + [a - b + \lambda(a + k\gamma)]x_{-1} \\ + [\beta + bk + k' + (\delta - a)\lambda k]x_1 \,,$$

where the functions P, Q are defined by

$$P = p + ba - b^2 + a\beta - b_u \,, \qquad Q = q + a\delta - a^2 + b\gamma - a_v \,.$$

This point is on the ray, so that P_y is the focal point of the ray corresponding to the curve C_λ, if, and only if, the functions λ and k are connected by the relation

$$P + K\lambda + Hk + Q\lambda k = 0 \,. \tag{29}$$

Moreover, the point P_y lies on the harmonic conjugate of the tangent of the curve C_λ with respect to the parametric tangents, i.e., on the tangent of the curve $C_{-\lambda}$ at P_x, in case $k = -\lambda$. Eliminating k from equation (29), and replacing λ by dv/du, we obtain the desired *differential equation of the net congruentially associated with the parametric net, namely,*

$$Pdu^2 - (H - K)du dv - Qdv^2 = 0 \,. \tag{30}$$

Let us now denote by y_1 the focal point of the ray that corresponds to the curve C_λ at the point P_x. The other curve of the congruentially associated net through the point P_x may be denoted by C_μ; the focal point of the ray corresponding to C_μ, and lying on the tangent of the curve $C_{-\mu}$ at P_x, may be denoted by y_2. Elimination of λ, instead of k as in the preceding paragraph, leads to the equation

$$Qk^2 - (H - K)k - P = 0 \,.$$

The two foci y_1, y_2 are determined by solving this equation for k_1, k_2 and substituting these two roots into the original expression for y.

Green's characterization of plane nets with equal invariants is stated only for such nets as have proper congruentially associated nets. For the parametric net of system (26) this restriction obviously is $(H-K)^2+4PQ\neq 0$. The statement of the characterization is as follows. *A plane net with a proper congruentially associated net has equal Laplace-Darboux invariants* $(H=K)$ *if, and only if, at each of its points the tangents of the congruentially associated net separate the tangents of the given net harmonically.* The demonstration is easily made by considering the harmonic, or simultaneous, invariant of the binary quadratic form in equation (30) and of the form $dudv$.

For a plane net with equal invariants the tangents of the curves C_μ and $C_{-\lambda}$ at the point P_x coincide, as do also the tangents of C_λ and $C_{-\mu}$. Consequently the tangent of C_μ passes through the focal point y_1 corresponding to C_λ, and the tangent of C_λ passes through the focal point y_2 corresponding to C_μ. Therefore a second form of Green's characterization may be stated as follows.

A plane net with a proper congruentially associated net has equal invariants if, and only if, at each of its points the tangent of each curve of the congruentially associated net passes through that focal point of the ray of the given net that corresponds to the other curve of the congruentially associated net.

30. Conjugate nets in ordinary space. This section and the next are devoted to the theory of conjugate nets in ordinary space. The emphasis in the present section is placed on the fundamental and general aspects of the theory, while consideration of special classes of nets is for the most part reserved for the following section. The completely integrable system of differential equations that define a conjugate net in ordinary space, except for a projective transformation, are written in a symmetrical form intimately connected with the axis congruence of the net. A canonical form of these equations suggests itself naturally and is used throughout the discussion. The developables and focal surfaces of the ray and axis congruences of a parametric conjugate net are determined in this section; the Weingarten invariants are introduced; and the point and tangential Laplace equations of a conjugate net are compared.

The first problem is to establish *the defining system of differential equations*. Let us consider in ordinary space S_3 a surface S with the parametric vector equation $x=x(u, v)$, referred to a conjugate net N_x, so that x satisfies equation (1). The osculating planes of the parametric curves C_u, C_v at a point P_x on the surface S intersect in the axis of P_x with respect to the net N_x. The axis does not lie in the tangent plane of S at P_x. Let P_y be any

point distinct from P_x on the axis. Then x and y satisfy a *system of differential equations* of the form

$$
\left\{
\begin{aligned}
x_{uu} &= px + ax_u + Ly \ , \\
x_{uv} &= cx + ax_u + bx_v \ , \\
x_{vv} &= qx + \delta x_v + Ny
\end{aligned}
\right.
\qquad (LN \neq 0) \ , \tag{31}
$$

as may be seen by observing that the first and third equations express the fact that the point P_y lies in the osculating planes of the curves C_u and C_v respectively at P_x, while the second is merely the equation (1) of Laplace.

We are going to *choose for the point P_y the harmonic conjugate of the point P_x with respect to the two foci of the axis.* In order to find these foci we observe that the point P_z defined by

$$z = y + kx \qquad (k \text{ scalar})$$

is on the axis. When the point P_x varies along a curve C_λ of the family $dv - \lambda du = 0$ on the surface S, the point P_z generates a curve whose tangent at P_z is determined by P_z and the point z' given by

$$z' = y_u + y_v\lambda + k(x_u + x_v\lambda) + k'x \ .$$

In order to calculate expressions for the derivatives y_u and y_v as linear combinations of x, x_u, x_v, y, it is sufficient to equate the two expressions for x_{vvu}, and the two for x_{uuv}, derivable from equations (31). Thus we find

$$
\left\{
\begin{aligned}
y_u &= fx + mx_u + sx_v + Ay \ , \\
y_v &= gx + tx_u + nx_v + By \ ,
\end{aligned}
\right. \tag{32}
$$

where we have placed

$$
\left\{
\begin{aligned}
fN &= c_v + ac + bq - c\delta - q_u \ , & gL &= c_u + bc + ap - c\alpha - p_v \ , \\
mN &= a_v + a^2 - a\delta - q \ , & tL &= a_u + ab + c - \alpha_v \ , \\
sN &= b_v + ab + c - \delta_u \ , & nL &= b_u + b^2 - b\alpha - p \ , \\
A &= b - (\log N)_u \ , & B &= a - (\log L)_v \ .
\end{aligned}
\right. \tag{33}
$$

Expressing z' as a linear combination of x, x_u, x_v, y, and equating to zero the coefficient of x_u, x_v therein, we obtain conditions on the functions k, λ necessary and sufficient that the axis may generate a developable and have P_z for focal point when P_x varies along C_λ, namely,

$$m + k + t\lambda = 0 \ , \qquad s + (n+k)\lambda = 0 \ .$$

Elimination of λ gives

$$k^2+(m+n)k+mn-st=0\ . \tag{34}$$

If k_1, k_2 are the roots of this equation the corresponding points z_1, z_2 are the foci of the axis. The point y is the harmonic conjugate of the point x with respect to these foci in case $m=-n$. *We shall suppose from now on that* $m=-n$. Incidentally, on eliminating k and replacing λ by dv/du, we obtain *the differential equation of the axis curves of the net* N_x, in which the developables of the axis congruence intersect the surface S, namely,

$$sdu^2+2ndudv-tdv^2=0\ . \tag{35}$$

The tangents of these curves are called *the axis tangents* of the net N_x. At the close of this section we shall write equation (35) in another form.

The integrability conditions of system (31) are found by equating the two expressions for y_{uv} derivable from equations (32), and by reducing the result to a linear equation in x, x_u, x_v, y, all of whose coefficients must vanish. Thus one obtains

$$\left\{\begin{aligned} &t_u+n_v+ta+2an+g=tA+nB\ ,\\ &s_v-n_u+s\delta-2bn+f=sB-nA\ ,\\ &f_v+qs+gA=g_u+pt+fB+2cn\ ,\\ &\qquad A_v+sN=B_u+tL\ . \end{aligned}\right. \tag{36}$$

Certain consequences of these integrability conditions will be needed later on. The result of substituting the values of s, t from equations (33) into the last of these conditions is

$$(b+a+A)_v=(a+\delta+B)_u\ ; \tag{37}$$

the result of substituting into this equation the values of A, B from equations (33) is

$$2(H-K)-(\log r)_{uv}=\delta_u-a_v \qquad (r=N/L)\ . \tag{38}$$

The ratio r introduced here will occur again frequently.

By means of equations (II, 9), (II, 10) *the differential equation of the asymptotic curves on the surface* S is easily shown to

$$Ldu^2+Ndv^2=0\ . \tag{39}$$

In order to apply the same equations (II, 9), (II, 10) to the surface S_1 sustaining the first Laplace transformed net N_1, we start from the first of equations (2) and find by direct calculations

$$(40) \quad \begin{cases} x_{1\,u} = bx_1 + Hx\ , & x_{1\,v} = (\delta - a)x_1 + N(nx+y)\ , \\ x_{1\,uu} = (bx_1)_u + H_u x + Hx_u\ , & x_{1\,uv} = c_1 x_1 + a_1 x_{1\,u} + b_1 x_{1\,v}\ , \\ x_{1\,vv} = [(\delta-a)x_1]_v + 2nNx_1 + tNx_u + [nN_v + N(n_v + 2an + g)]x & \\ \qquad + (N_v + BN)y\ . & \end{cases}$$

Indeed, the first of these equations is the same as equation (18), and the coefficients of the fourth are given in equations (19). *The differential equation of the asymptotic curves on the surface* S_1 can now readily be shown to be

$$(41) \qquad Hdu^2 + tNdv^2 = 0\ .$$

Similarly, or else by means of the substitution

$$(42) \qquad \begin{pmatrix} u & a & c & f & p & s & n & \alpha & A & L & H & x_r \\ v & b & c & g & q & t & -n & \delta & B & N & K & x_{-r} \end{pmatrix},$$

the differential equation of the asymptotics on the surface S_{-1}, sustaining the Laplace transformed net N_{-1}, is found to be

$$(43) \qquad sLdu^2 + Kdv^2 = 0\ .$$

The asymptotic curves on the surfaces S and S_{-1} correspond in case equations (39) and (43) are equivalent. Then the u-tangents of the net N_x form a congruence of the special type called *a W congruence.* Thus we reach the result that *the u-tangents of the net* N_x *form a W congruence in case the Weingarten invariant* $W^{(u)}$, *defined by*

$$(44) \qquad W^{(u)} = sN - K\ ,$$

vanishes. Similarly, *the v-tangents of the net* N_x *form a W congruence in case the Weingarten invariant* $W^{(v)}$, *defined by*

$$(45) \qquad W^{(v)} = tL - H\ ,$$

vanishes. Substituting in equation (38) the values of δ_u, a_v given by equations (33), then eliminating s, t by means of the definitions (44), (45), and employing finally the definitions (7), we arrive at an important relation,

$$(46) \qquad W^{(u)} - W^{(v)} = (\log r)_{uv} \qquad (r = N/L)\ .$$

We shall next determine *the developables and focal surfaces of the ray congruence of the net N_x*. The point P_ζ defined by

$$\zeta = x_{-1} + kx_1 \qquad (k \text{ scalar})$$

is on the ray of N_x corresponding to the point P_x. When P_x varies along a curve C_λ of the family $dv - \lambda du = 0$ on the surface S, the point P_ζ generates a curve whose tangent at P_ζ is determined by P_ζ and the point ζ' given by

$$\zeta' = x_{-1\,u} + x_{-1\,v}\lambda + k(x_{1\,u} + x_{1\,v}\lambda) + k'x_1 \; .$$

By use of the first two of equations (40) and the symmetric equations written by means of the substitution (42), the derivative ζ' can be expressed as a linear combination of x_{-1}, x_1, x, y. Equating to zero the coefficients of x, y in this expression we obtain conditions on the functions k, λ necessary and sufficient that the ray may generate a developable and have P_ζ for focal point when P_x varies along C_λ, namely,

$$-nL + K\lambda + Hk + nNk\lambda = 0 \; , \qquad L + Nk\lambda = 0 \; .$$

Elimination of λ gives

$$NHk^2 - 2nLNk - LK = 0 \; . \tag{47}$$

If k_1, k_2 are the roots of this equation, then the corresponding points ζ_1, ζ_2 are *the foci of the ray*. Moreover, elimination of k gives, on replacing λ by dv/du, *the differential equation of the ray curves of the net N_x*, which correspond on the surface S to the developables of the ray congruence, namely,

$$LHdu^2 + 2nLNdudv - NKdv^2 = 0 \; . \tag{48}$$

The tangents of these curves are called *the ray tangents* of the net N_x.

Another method of determining the foci of the ray yields also an interesting theorem. We find, by differentiation and substitution,

$$\left\{ \begin{aligned} \zeta_u &= (a-b)x_{-1} + (k_u + bk)x_1 + (Hk - nL)x + Ly \; , \\ \zeta_v &= ax_{-1} + [k_v + (\delta - a)k]x_1 + (K + nNk)x + Nky \; . \end{aligned} \right. \tag{49}$$

Let us consider any plane through the ray (containing therefore the points x_{-1}, x_1) and meeting the axis in a point $y + \mu x$. We wish this plane to be what is called *a focal plane* of the ray; so we impose the condition that it be

tangent at a point ζ to a focal surface of the ray congruence. This condition is simply that the plane contain the points ζ_u, ζ_v; thus we find

$$\mu=(Hk-nL)/L=(K+nNk)/Nk\ .$$

It follows that k must be a root of equation (47); in this way we have determined the foci of the ray again. If μ_1, μ_2 correspond respectively to the roots k_1, k_2 of equation (47), a simple calculation gives $\mu_1+\mu_2=0$. Thus we have proved* the following theorem:

At a point P_x of a surface sustaining a conjugate net N_x in ordinary space the harmonic conjugate of the tangent plane of the surface, with respect to the two focal planes of the ray of N_x, passes through the point y which is the harmonic conjugate of P_x with respect to the two foci of the axis of N_x.

Equation (1) is sometimes called *the point equation* of the net N_x. The coordinates of the tangent plane of the surface S sustaining N_x also satisfy an equation of Laplace which is called *the tangential equation* of N_x (see Ex. 19). In order to find this equation in a simple form, we observe that equation (37) shows that there exists a function θ defined, except for an additive constant, by the differential equations

$$\theta_u=b+a+A\ , \qquad \theta_v=a+\delta+B\ .$$

If the coordinates ξ of the tangent plane at a point P_x of the surface S are defined by the formula

$$\xi=e^{-\theta}(x,\ x_u,\ x_v)\ , \tag{50}$$

then direct calculation shows that ξ satisfies the Laplace equation

$$\xi_{uv}=(tL-AB-A_v)\xi-B\xi_u-A\xi_v\ . \tag{51}$$

This is *the tangential equation* of the net N_x.

Sometimes the Laplace-Darboux invariants H, K of equation (1) are called *point invariants* of the net N_x, while the corresponding invariants $\mathscr{H}$, $\mathscr{K}$ of equation (51) are called *tangential invariants* of N_x. These invariants and the Weingarten invariants defined by equations (44), (45) are connected by the relations

$$\mathscr{H}=K+W^{(u)}\ , \qquad \mathscr{K}=H+W^{(v)}\ . \tag{52}$$

* Slotnick, 1931. 1, p. 148.

Consequently the equation (35) of the axis curves can now be rewritten in the form

$$L\mathscr{H}du^2+2nLNdudv-N\mathscr{K}dv^2=0 \ .$$

Moreover, the equations (41), (43) of the asymptotic curves on the surfaces S_1, S_{-1} become respectively

$$LHdu^2+N\mathscr{K}dv^2=0 \ , \qquad L\mathscr{H}du^2+NKdv^2=0 \ .$$

31. Special classes of conjugate nets in ordinary space. There are certain classes of conjugate nets distinguished by possessing special properties. One of the most interesting of these properties is that of isothermal conjugacy, and so the class of *isothermally conjugate nets* will be the first to claim our attention in this section. Two other important classes are the class of *harmonic nets* and the class of *R nets*. These and other classes will be discussed, and incidentally the theory of an unspecialized net will be amplified to some extent, for example, by the calculation of the differential equations of the surface sustaining the first Laplace transformed net.

The property of *isothermal conjugacy* of a net of curves on a surface in ordinary space was first defined* by Bianchi. His definition was of an analytic nature; although it differs in form from the definition stated in Exercise 26 of Chapter III, the two definitions are in fact equivalent. Bianchi showed that the property of isothermal conjugacy is of a projective character, but he did not give a purely geometrical description of it. No serious attempt seems to have been made to discover such a description until 1915 when† Wilczynski found an algebraic relation between certain geometrically interpreted projective invariants which is characteristic of isothermally conjugate nets. In 1916 Green gave‡ the purely geometric description embodied in Exercise 8, which is now known to be characteristic only of *non-harmonic* isothermally conjugate nets, although Green mistakenly supposed at first that he had completely solved the problem of characterizing isothermally conjugate nets geometrically. Wilczynski was the first to reach a complete solution§ of the problem. His final solution is embodied in Exercise 26 of Chapter III, and was given‖ its present analytically simple form by the author.

We now undertake to find an analytic condition that the parametric conjugate net N_x of the present chapter be isothermally conjugate. For this purpose we make a transformation of curvilinear coordinates from the

* Bianchi, 1922. 5, p. 210.

† Wilczynski, 1915. 1, p. 323.

‡ Green, 1916. 1, p. 313.

§ Wilczynski, 1920. 1, p. 221.

‖ Lane, 1922. 3, p. 292.

asymptotic coordinates u, v of Chapter III to the conjugate coordinates of the present chapter, which will be denoted in just this paragraph by ξ, η. Such a transformation is represented by

$$\xi=\xi(u,\,v)\ ,\qquad \eta=\eta(u,\,v)\qquad\qquad (\xi_u\eta_v-\xi_v\eta_u\neq 0)\ ,$$

and differentiation gives

$$d\xi=\xi_u du+\xi_v dv\ ,\qquad d\eta=\eta_u du+\eta_v dv\ . \tag{53}$$

Now the net $d\xi d\eta=0$ is to be the same as the net $dv^2-\lambda^2du^2=0$ of Chapter III; it follows that

$$\xi_u\eta_v+\xi_v\eta_u=0\ ,\qquad \xi_u\eta_u+\lambda^2\xi_v\eta_v=0\ .$$

These conditions can be satisfied by taking ξ, η as solutions of the equations

$$\xi_u=-\lambda\xi_v\ ,\qquad \eta_u=\lambda\eta_v\ , \tag{54}$$

from which one obtains, by differentiation and substitution,

$$\begin{cases}\xi_{uv}=-\lambda_v\xi_v-\lambda\xi_{vv}\ , & \xi_{uu}=-(\lambda_u-\lambda\lambda_v)\xi_v+\lambda^2\xi_{vv}\ ,\\ \eta_{uv}=\lambda_v\eta_v+\lambda\eta_{vv}\ , & \eta_{uu}=(\lambda_u+\lambda\lambda_v)\eta_v+\lambda^2\eta_{vv}\ .\end{cases} \tag{55}$$

Solution of equations (53) for du, dv, elimination of ξ_u, η_u by (54), and subsequent multiplication result in

$$-4\lambda\xi_v^2\eta_v^2dudv=\eta_v^2d\xi^2-\xi_v^2d\eta^2\ .$$

Therefore the differential equation of the asymptotic curves in conjugate parameters is

$$\eta_v^2d\xi^2-\xi_v^2d\eta^2=0\ . \tag{56}$$

Comparison of this equation with equation (39) shows that

$$r=-\xi_v^2/\eta_v^2\qquad\qquad (r=N/L)\ .$$

As in calculating equations (12) we find, on differentiating any function x of u, v,

$$x_u=x_\xi\xi_u+x_\eta\eta_u\ ,\qquad x_v=x_\xi\xi_v+x_\eta\eta_v\ .$$

Solution of these two equations for x_ξ and x_η furnishes two differentiation formulas,

$$(57)\qquad x_\xi=-(x_u-\lambda x_v)/2\lambda\xi_v\ ,\qquad x_\eta=(x_u+\lambda x_v)/2\lambda\eta_v\ ,$$

by the aid of which we obtain by direct calculation* of the derivatives of r,

$$(58)\qquad \lambda\xi_v\eta_v(\log r)_{\xi\eta}=(\log\lambda)_{uv}\ .$$

In Exercise 26 of Chapter III the defining condition for isothermal conjugacy was $(\log\lambda)_{uv}=0$. Our result may therefore be stated as follows:

A parametric conjugate net $d\xi d\eta=0$ is isothermally conjugate if, and only if, $(\log r)_{\xi\eta}=0$.

The form of system (31) is left unchanged by every transformation of the group

$$(59)\qquad x=\lambda(u,v)\bar{x}\ ,\qquad y=\mu(u,v)\bar{y}\ ,\qquad \bar{u}=U(u)\ ,\qquad \bar{v}=V(v)\qquad (\lambda\mu U'V'\neq 0)\ .$$

The effect of such a transformation on the coefficients of system (31) is easily calculated (see Ex. 25). In particular, we find that the effect of the transformation $x=\lambda\bar{x}$ on the coefficients α, δ is given by

$$(60)\qquad \bar{\alpha}=\alpha-2\,(\log\lambda)_u\ ,\qquad \bar{\delta}=\delta-2\,(\log\lambda)_v\ ,$$

but we shall not need to write the formulas for the other coefficients. Similarly, the effect of the transformation $\bar{u}=U(u)$, $\bar{v}=V(v)$ on the coefficients L, N is found to be given by

$$(61)\qquad \bar{L}=L/U'^2\ ,\qquad \bar{N}=N/V'^2\ .$$

If the parametric net $dudv=0$ is isothermally conjugate, then $(\log r)_{uv}=0$ and hence $r=U_1(u)V_1(v)$ where U_1, V_1 are arbitrary functions of the arguments indicated. It follows from equations (61) and the definition of r in (38) that it is possible to make a transformation of parameters so that after the transformation we shall have $r=-1$. In fact, it is sufficient to use functions U, V satisfying the conditions $U'^2U_1=-V'^2/V_1=\text{const.}$ Thus we reach the conclusion:

It is no restriction on an isothermally conjugate net to suppose that the parameters have been chosen so that $r=-1$.

Another special class of conjugate nets consists of harmonic nets. *A conjugate net for which $n=0$ is called† a harmonic net* because it possesses a num-

* *Ibid.*, p. 288.

† Wilczynski, 1920. 1, p. 215.

ber of harmonic properties. Some of these properties will now be indicated, under the supposition that the harmonic conjugate net considered is not further specialized. Equation (35) shows that *at each point of a harmonic net the axis tangents separate the tangents of the net harmonically.* Moreover, since the axis tangents and the axis determine the focal planes of the axis, it follows that *at each point of a harmonic net the focal planes of the axis separate the osculating planes of the net harmonically.* Equation (48) shows that *at each point of a harmonic net the ray tangents separate the tangents of the net harmonically,* while equation (47) shows that *the foci of the ray separate the ray points harmonically.* Finally, the second of equations (40) and the symmetric equation that can be written by means of the substitution (42) show that *at each point P_x of a harmonic net N_x the point y, which is the harmonic conjugate of P_x with respect to the foci of the axis, is also the point of intersection of the v-tangent at the corresponding point x_1 of the net N_1* (which joins the points x_1, x_2) *and the u-tangent at the corresponding point x_{-1} of the net N_{-1}* (which joins the points x_{-1}, x_{-2}).

It is perhaps worthy of remark that two corresponding points P_x, P_y *for an unspecialized conjugate net N_x* are not only separated harmonically by foci $y \pm (n^2 + \mathscr{H}\mathscr{K}/LN)^{1/2}x$ of the axis of P_x, but also by the points $y \pm nx$ where the v-tangent of the net N_1 and the u-tangent of the net N_{-1} meet the axis, as well as by the points $y \pm (n^2 + HK/LN)^{1/2}x$ where the focal planes of the corresponding ray intersect the axis. The point $y + nx$ could also be defined as the point where the osculating plane of the v-curve of the net N_x is tangent to the ruled surface of axes constructed at the points of the u-curve through the point P_x, and there is a symmetric definition for the point $y - nx$.

A third special class of conjugate nets consists of R nets. *A conjugate net for which $W^{(u)} = W^{(v)} = 0$ is called* an R net;* each family of curves of an R net have tangents that form a W congruence. Inspection of equation (46) shows that *an R net is isothermally conjugate,* and that *an isothermally conjugate net such that the tangents of the curves of one family form a W congruence is an R net.*

Some of the facts concerning *ray curves and axis curves that form conjugate nets* are as follows. Calculation of the harmonic invariant of the binary quadratic forms in equations (39) and (48) shows that *the ray curves of a conjugate net themselves form a conjugate net*† *if, and only if, $H = K$.* Similarly, *the axis curves form a conjugate net in case $\mathscr{H} = \mathscr{K}$.* Furthermore, *if the ray and axis curves of a conjugate net both form conjugate nets, the fundamental*

* Tzitzéica, 1911. 1.

† Wilczynski, 1915. 1, p. 319.

net is isothermally conjugate. Moreover, *an isothermally conjugate net has ray curves forming a conjugate net if, and only if, its axis curves also form a conjugate net.* For such a net we find from equation (38) that $\delta_u - \alpha_v = 0$; when this condition is satisfied, equations (60) show that by a transformation of proportionality factor it is possible to make $\alpha = \delta = 0$.

Investigation of the possibilities as to *coincidence of the ray curves and axis curves* yields the following results. *The ray curves and axis curves of a non-harmonic conjugate net coincide in case* $H = \mathscr{H}, K = \mathscr{K}$. If, in addition, the fundamental net is isothermally conjugate, then the ray-and-axis curves form a conjugate net, and the fundamental net is an R net. Furthermore, *if a non-harmonic conjugate net has* $HK \neq 0$, *the ray curves and axis curves coincide in case the developables of the ray congruence intersect the surfaces* S_1, S_{-1} *in conjugate nets. The ray curves and axis curves of a harmonic net coincide in case* $H\mathscr{K} - K\mathscr{H} = 0$. If, in addition, the fundamental net is isothermally conjugate, then either the ray-and-axis curves form a conjugate net or else the fundamental conjugate net is restricted by the conditions $H + \mathscr{H} = 0$, $K + \mathscr{K} = 0$.

The following statements concerning *the ray tangents and axis tangents* are of some interest. Equations (47), (48) show that *the ray tangents at each point of a non-harmonic conjugate net meet the corresponding ray in the focal points of the ray if, and only if, the ray curves form a conjugate net.* The ray tangents at each point of a harmonic net meet the corresponding ray in the focal points of the ray under either of two hypotheses; first, if the ray curves form a conjugate net; second, if the fundamental net is restricted by the condition $H + K = 0$, i.e., if the ray curves coincide with the asymptotic curves. *The axis tangents at each point of a non-harmonic net meet the corresponding ray in the focal points of the ray, if, and only if, the net is an R net.* The axis tangents at each point of a harmonic net meet the corresponding ray in the focal points of the ray in case the fundamental net is restricted by the condition $H\mathscr{H} = K\mathscr{K}$.

Referring to Figure 21, we propose to find a system of equations of the form (31) for the first Laplace transformed surface S_1 of an integral surface S of system (31). The second equation of the system for the surface S_1, namely, the Laplace equation, was found in Section 28, its coefficients being given in equations (19). In order to find the other two equations let us observe that the point X defined by

$$X = x_u + [A - \alpha - (\log t)_u]x \tag{62}$$

is certainly on the u-tangent at the point P_x of the parametric net N_x on the surface S. Moreover, the point X is also on the axis of the point x_1

with respect to the net N_1 on the surface S_1, since the point X can be shown by equations (40) and the first of the integrability conditions (36) to lie in

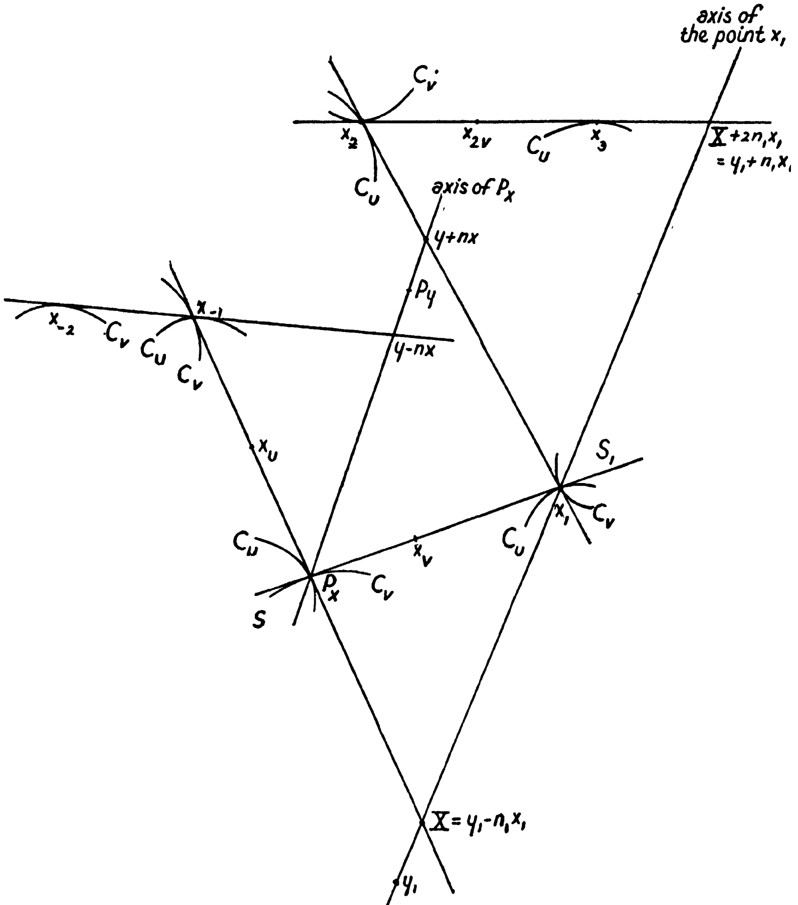

FIG. 21

the osculating planes at the point x_1 of the curves C_u and C_v of the net N_1. Furthermore, the point

$$X+2n_1x_1\,, \tag{63}$$

where n_1 is defined by placing

$$(64) \qquad 2r\mathcal{K}n_1=(\delta-a-a_1)_v+(\delta-a-a_1)(\log H/r)_v+2nN \ ,$$

is certainly on the axis of the point x_1, and can be shown by means of equations (40) and the first of the integrability conditions (36) to be also on the v-tangent of the net N_2. In fact, the point (63) is identical with the point

$$x_{2\,v}+[\delta-a-(\log H/r)_v]x_2 \ .$$

The harmonic conjugate y_1 of x_1 with respect to X and $X+2n_1x_1$ is given by

$$(65) \qquad y_1=X+n_1x_1 \ .$$

That the function n_1 is, as the notation indicates, the coefficient n for the net N_1 is verified by observing that the points X and $X+2n_1x_1$ can be expressed in the respective forms

$$(66) \qquad y_1-n_1x_1 \ , \qquad y_1+n_1x_1 \ .$$

Now that we know the expression for y_1 the rest of the calculation for the required system of equations for the surface S_1 is largely mechanical. Use is made of equations (33), (40), (62), (65). The result is a system of equations of the form (31) whose coefficients, indicated by the appropriate subscript, are given by the following formulas:

$$(67) \qquad \begin{cases} p_1=p+nN-n_1H-b(\log rH\mathcal{K})_u \ , \\ a_1=a+(\log rH\mathcal{K})_u \ , \qquad L_1=H \ , \\ c_1=c+H-K-b(\log H)_v \ , \qquad a_1=a+(\log H)_v \ , \qquad b_1=b \ , \\ q_1=(\delta-a)_v-(\delta-a)[a+(\log r)_v]+2nN-rn_1\mathcal{K}, \\ \delta_1=\delta+(\log \mathrm{r})_v \ , \qquad N_1=r\mathcal{K}. \end{cases}$$

As an application of these formulas, let us calculate the Weingarten invariants $W_1^{(u)}$, $W_1^{(v)}$ and the invariant r_1 for the net N_1. We find

$$(68) \qquad W_1^{(u)}=W^{(v)} \ , \qquad W_1^{(v)}=W^{(v)}-(\log r_1)_{uv} \ , \qquad r_1=r\mathcal{K}/H \ .$$

It is easy to see that, if $W^{(u)}=W^{(v)}=0$, then $W_1^{(u)}=W_1^{(v)}=0$. Thus we prove the theorem:

A Laplace transform of an R net is an R net.

32. Conjugate and harmonic relations of nets and congruences. When the points of a conjugate net and the lines of a congruence in space S_n are in one-to-one correspondence, there are two special relative positions of the net and congruence which are of particular interest. In one of these positions the net and congruence are said* to be *conjugate* to each other, and in the second *harmonic* to each other. These relations of net and congruence will be explained more precisely presently, and then it may be observed that they are dual relations in the dualistic transformation that converts each point of a net into the tangent plane at this point of the surface sustaining the net, which may be called simply *the tangent plane of the net.*

Various properties of the conjugate and harmonic relations of nets and congruences in space S_n will be studied in this section. The problems of determining analytically all nets conjugate, or harmonic, to a given congruence will be solved, as well as the problems of determining all congruences conjugate, or harmonic, to a given net. The correspondence between two nets known as *the Levy transformation* will be introduced.

The conjugate relation of a net and a congruence is defined as follows. *A net and a congruence, such that there is just one line of the congruence through each point of the net and not in the tangent plane of the net at the point, are conjugate to each other in case the developables of the congruence intersect the surface sustaining the net in the curves of the net.* A focal net of a congruence cannot be conjugate to the congruence. A bundle of lines, such that there is just one of its lines through each point of a net and not in the tangent plane at the point, is said to be conjugate to the net.

We shall now determine *all nets that are conjugate to a given congruence.* Any congruence $\Gamma_{\eta\zeta}$ with non-degenerate focal nets N_η, N_ζ in space S_n can

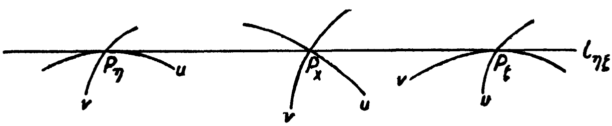

FIG. 22

be represented analytically by writing the equations which assert that the *u-tangent* at a point η is also the v-tangent at a point ζ, as in Figure 22. These equations take the form

$$\eta_u = \alpha\eta + \beta\zeta , \qquad \zeta_v = \gamma\eta + \delta\zeta \qquad (\beta\gamma \neq 0) . \tag{69}$$

Indeed, by a transformation $\eta = \lambda\bar{\eta}$, $\zeta = \mu\bar{\zeta}$ it is possible to reduce each of α, δ to zero, but we shall not make use of this simplification for the present.

* Guichard, 1897. 2, pp. 478 and 483.

Elimination of ζ from equations (69) furnishes the Laplace equation for the net N_η, namely,

$$\eta_{uv}=[\beta\gamma-a\delta+a_v-a(\log\beta)_v]\eta+[\delta+(\log\beta)_v]\eta_u+a\eta_v\ , \tag{70}$$

while elimination of η would result in the Laplace equation for N_ζ. Any point P_x on the line $l_{\eta\zeta}$ can be defined by placing

$$x=\mu\eta-\lambda\zeta \qquad (\lambda,\ \mu \text{ scalars})\ . \tag{71}$$

If the functions λ, μ satisfy equations (69), i.e., if

$$\lambda_u=a\lambda+\beta\mu\ , \qquad \mu_v=\gamma\lambda+\delta\mu\ , \tag{72}$$

it can be shown that the point P_x generates a net N_x conjugate to the congruence $\Gamma_{\eta\zeta}$. In fact, differentiation of equation (71), followed by judicious use of (69), (72), leads to

$$x_u-ax=\mu_u\eta-\lambda\zeta_u\ , \qquad x_v-\delta x=\mu\eta_v-\lambda_v\zeta\ . \tag{73}$$

Differentiation of the second of these equations with respect to u, and subsequent use of equations (69), . . . , (73), lead finally to a Laplace equation for the net N_x,

$$\left\{\begin{aligned} x_{uv}=[a_v+\delta_u+\beta\gamma-a\delta-a(\log\lambda)_v-\delta(\log\mu)_u-(\log\lambda)_v(\log\mu)_u]x\\ +[\delta+(\log\lambda)_v]x_u+[a+(\log\mu)_u]x_v\ . \end{aligned}\right. \tag{74}$$

Conversely, if the point P_x defined by equation (71) generates a net conjugate to the congruence $\Gamma_{\eta\zeta}$, then the functions λ, μ can be multiplied by such a common factor, without changing P_x, that they satisfy equations (72). The proof* will be omitted. Thus we arrive at the conclusion:

A congruence $\Gamma_{\eta\zeta}$ *represented by equations* (69) *is conjugate to every net* N_x *defined by equation* (71) *subject to the conditions* (72), *and every net* N_x *conjugate to the congruence* $\Gamma_{\eta\zeta}$ *can be so represented.*

The harmonic relation of a net and a congruence is defined as follows. *A net and a congruence, such that there is just one line of the congruence in each tangent plane of the net and not through the point of contact, are harmonic to each other in case the developables of the congruence correspond to the curves of the net.* A focal net of a congruence cannot be harmonic to the congruence. Just as the focal planes of a generator of a congruence conjugate to a net contain the tangents of the net at the point on the generator, so the focal

* Tzitzéica, 1924. 3, p. 69.

points of a generator of a congruence harmonic to a net lie on the tangents of the net that are in the tangent plane containing the generator.

The problem of determining all congruences that are harmonic to a given net can be solved in the following way. Let us consider a net N_x for which equation (1) is valid. The points P_ρ, P_σ defined by

$$\rho = x_u + hx\ , \qquad \sigma = x_v + kx \qquad (h,\ k \text{ scalars}) \tag{75}$$

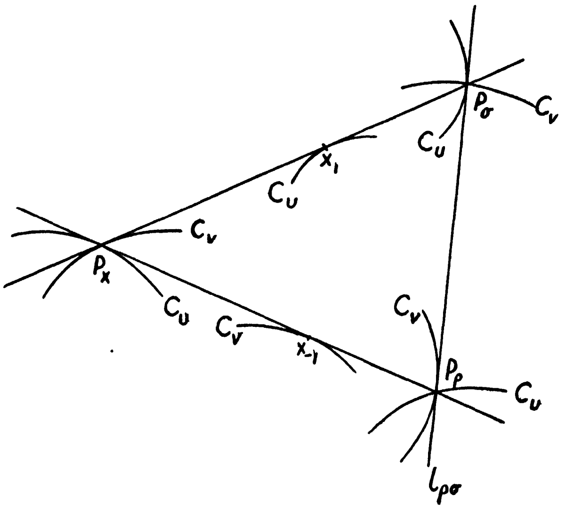

FIG. 23

are on the tangents of the curves C_u, C_v respectively at the point P_x as indicated in Figure 23. Simple calculation leads to

$$\begin{cases} \rho_v = a\rho + (b+h)\sigma + (h_v + c - ah - bk - hk)x\ , \\ \sigma_u = (a+k)\rho + b\sigma + (k_u + c - ah - bk - hk)x\ . \end{cases} \tag{76}$$

If the line $l_{\rho\sigma}$ generates a developable surface as u varies, with P_σ as focal point, the coefficient of x in the second of equations (76) must vanish; similarly, if $l_{\rho\sigma}$ generates a developable as v varies, with P_ρ as focal point, the coefficient of x in the first of (76) must vanish. Therefore *every congruence harmonic to a net N_x with x satisfying equation* (1) *can be generated by the line $l_{\rho\sigma}$ joining the points P_ρ, P_σ defined by equations* (75) *with the functions h, k satisfying the conditions*

$$h_v = k_u = ah + bk + hk - c\ . \tag{77}$$

These points P_σ and P_ρ will be called respectively *the first and second focal points* of the line $l_{\rho\sigma}$.

It is easy to show that x *can be multiplied by such a factor that any congruence harmonic to the net* N_x *will be generated by the line joining the points* x_u, x_v. This amounts to showing that by a transformation $x=\lambda\bar{x}$ it is possible to reduce both of h, k to zero. We find

$$\bar{\rho}=\lambda\bar{x}_u+(\lambda_u+h\lambda)\bar{x}\ , \qquad \bar{\sigma}=\lambda\bar{x}_v+(\lambda_v+k\lambda)\bar{x}\ .$$

It is sufficient for our purpose to choose the function λ as a solution of

$$(\log \lambda)_u+h=0\ , \qquad (\log \lambda)_v+k=0\ ,$$

and this choice is possible because $h_v=k_u$. After this simplification has been made, equation (77) reduces to $c=0$. This is, in a sense, the simplest analytic representation of a congruence harmonic to a net.

The transformation known as *the Levy transformation* is defined as follows. *A Levy transformation between two nets is the correspondence between a focal net of a congruence and any net conjugate to the congruence.* Formulas for Levy transforms* of a net N_x with x satisfying equation (1) will now be deduced. The equations of the form (69) for the congruence of u-tangents of the net N_x are

$$x_u=bx+x_{-1}\ , \qquad x_{-1\,v}=Kx+ax_{-1}\ , \tag{78}$$

and so the most general net conjugate to this congruence is generated by the point ρ defined by

$$\rho=\theta_{-1}x-\theta x_{-1}\ , \tag{79}$$

where the functions θ, θ_{-1} satisfy

$$\theta_u=b\theta+\theta_{-1}\ , \qquad \theta_{-1\,v}=K\theta+a\theta_{-1}\ . \tag{80}$$

The function θ is easily shown to be a solution of equation (1), and the notation θ_{-1} is intended to be suggestive. Therefore *any Levy transformed net* N_ρ *of the net* N_x *along the congruence of u-tangents of* N_x *is generated by the point* ρ *given by*

$$\rho=\theta_u x-\theta x_u\ , \tag{81}$$

* Levy, 1886. 2, p. 67.

where θ is a solution of the point equation of the net N_x. Similarly, any Levy transform N_σ of the net N_x along the v-tangents of N_x is generated by the point σ given by

$$\sigma = \theta_v x - \theta x_v \,, \tag{82}$$

where θ is a solution of the point equation of N_x.

If in equations (81), (82) we take the same function θ and then place

$$h = -(\log \theta)_u \,, \qquad k = -(\log \theta)_v \,,$$

we can easily show that these expressions for h, k satisfy equations (77). Therefore *two Levy transforms of a net along the two congruences of tangents of the net by means of the same solution of the point equation of the net are the focal nets of a congruence harmonic to the net.* Conversely, any congruence harmonic to a net can be so regarded.

It is easy to demonstrate that *if a net is conjugate to a congruence, one focal net of the congruence is harmonic to one of the congruences of tangents of the first net and the other focal net is harmonic to the other congruence of tangents.* For this purpose let us consider a congruence $\Gamma_{\eta\zeta}$ represented by equations (69), and a conjugate net N_x defined by equation (71) subject to the conditions (72). The first of equations (69) says that the u-tangent at the point P_η passes through the point P_ζ, as shown in Figure 24, and hence through the point P_x. Since we have

$$x_1 = x_v - [\delta + (\log \lambda)_v]x = \mu[\eta_v - (\log \lambda)_v \eta] \,, \tag{83}$$

it follows that the v-tangent at P_η passes through the point x_1. The demonstration that the congruence of v-tangents of the net N_x is harmonic to N_η is completed by observing that $x_{1\,u}$ is a linear combination of x_1 and x. Similarly, it can be demonstrated that the congruence of u-tangents of the net N_x is harmonic to the net N_ζ.

Two nets that are conjugate to one congruence are harmonic to another congruence. For, in the first place, it is geometrically clear that the tangent planes of the two nets at corresponding points, that is, points on the same generator of the common conjugate congruence, intersect in a line, since the tangent lines of corresponding curves at corresponding points of the two nets are coplanar. Analytic expressions for the coordinates of the two points

in which corresponding tangent lines intersect can be found in the following way. Let us consider a congruence $\Gamma_{\eta\zeta}$ represented by equations (69), and

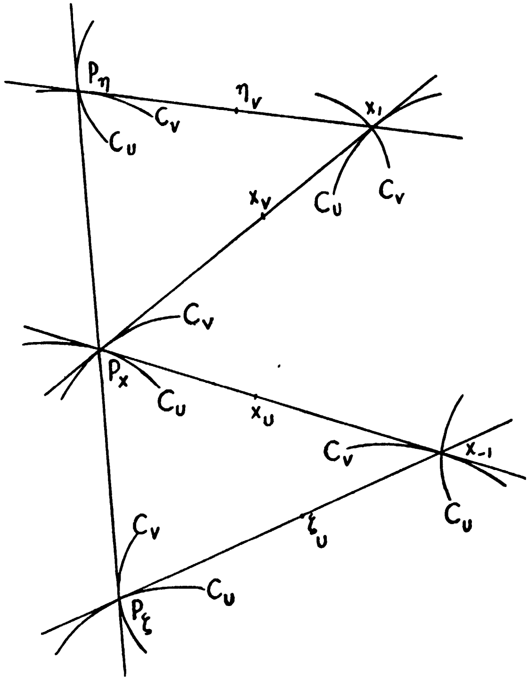

FIG. 24

a net N_x defined by equation (71) subject to the conditions (72), together with a net N_y defined by

$$y = \mu'\eta - \lambda'\zeta \tag{84}$$

with λ', μ' also satisfying (72), it being supposed that the determinant D defined by

$$D = \mu\lambda' - \mu'\lambda \tag{85}$$

is not zero. Simultaneous solution of equations (71) and (84) for η, ζ gives

$$\lambda' x - \lambda y = D\eta\,, \qquad \mu' x - \mu y = D\zeta\;; \tag{86}$$

and from the first of equations (73) together with the same equation with y, λ', μ' in place of x, λ, μ respectively we find, by elimination of ζ_u, the relation

$$\lambda'(x_u - ax) - \lambda(y_u - ay) = (\mu_u\lambda' - \mu'_u\lambda)\eta \tag{87}$$

Comparison of the two expressions for η in (87) and the first of equations (86) gives, after some reduction, the first of the following equations, the second being deduced similarly, or written by symmetry:

$$\begin{cases} \lambda'[x_u - (\log D)_u x] = \lambda[y_u - (\log D)_u y] , \\ \mu'[x_v - (\log D)_v x] = \mu[y_v - (\log D)_v y] . \end{cases} \tag{88}$$

Let us place

$$\begin{cases} x' = x/D , \quad y' = y/D , \\ m = \lambda/\lambda' , \quad n = \mu/\mu' \qquad (m - n = -D/\lambda'\mu' \neq 0) . \end{cases} \tag{89}$$

Then equations (88) become

$$x'_u = my'_u , \qquad x'_v = ny'_v . \tag{90}$$

The u-tangents at the points P_x, P_y intersect at the point x'_u, and the v-tangents at x'_v. From equations (90), on placing $\rho = y'_u$, $\sigma = y'_v$ and calculating $(x'_u/m)_v = (x'_v/n)_u$, we obtain the equations

$$(m-n)\rho_v = (m-n)\sigma_u = -m_v\rho + n_u\sigma ,$$

which tell us that the line $l_{\rho\sigma}$ generates a congruence harmonic to both of the nets N_x and N_y. Thus the demonstration is completed. Figure 25 illustrates the situation in the foregoing discussion.

We shall next determine *all nets that are harmonic to a given congruence.* For this purpose let us consider a congruence $\Gamma_{\eta\zeta}$ represented by equations (69), and let us seek to determine all nets N_x such that x satisfies two equations of the form

$$x_u = px + r\zeta , \qquad x_v = qx + s\eta , \tag{91}$$

which assert that the u-tangent at the point x passes through the point ζ, and that the v-tangent at the point x passes through the point η. Equating the two expressions for x_{uv} derivable from (91) leads, after replacing

x_u, x_v, η_u, ζ_v by their values, to a linear equation in x, η, ζ, all of whose coefficients must vanish. Thus one obtains for equations (91) three in-

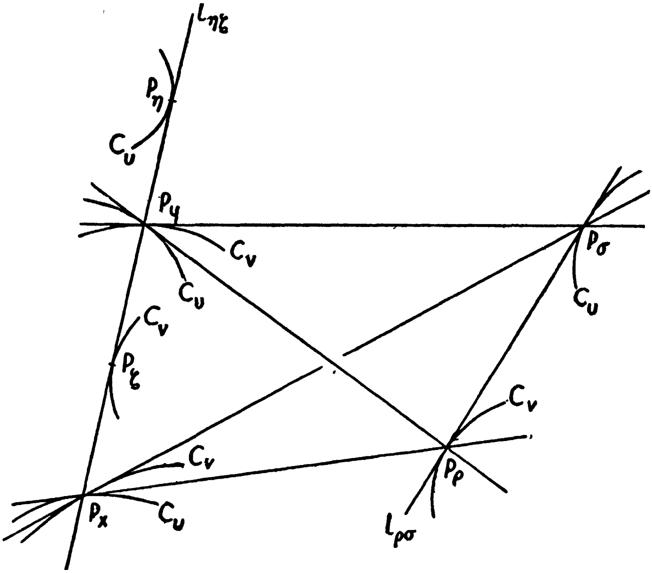

FIG. 25

tegrability conditions, one of which is $p_v = q_u$. Consequently a transformation $x = \lambda\bar{x}$ with

$$\lambda_u - p\lambda = \lambda_v - q\lambda = 0$$

can be made which will reduce both of p and q to zero. Then equations (91) become

$$x_u = r\zeta\ , \qquad x_v = s\eta\ , \tag{92}$$

The two integrability conditions for these equations are

$$r_v = -\delta r + \beta s\ , \qquad s_u = \gamma r - \alpha s\ . \tag{93}$$

Our conclusion may be stated as follows. *Every net N_x harmonic to the congruence $\Gamma_{\eta\zeta}$ represented by equations* (69) *is determined by equations* (92) *subject to the conditions* (93). We remark that to every pair of solutions r, s of (93) there correspond infinitely many nets N_x harmonic to $\Gamma_{\eta\zeta}$, since x is

determined by equations (92) except for an additive constant, when a pair of solutions r, s is known. When $r=s=0$ the point x is fixed and does not generate a proper net.

It is now easy to show that *two nets that are harmonic to one congruence are conjugate to another congruence.* To this end let us consider two nets N_x, N_y, the first being given by equations (92) subject to the conditions (93) and the second by the same equations with r', s' in place of r, s. Both of these nets are harmonic to the congruence $\Gamma_{\eta\zeta}$ represented by equation (69). We suppose $rsr's'\neq 0$ and find

$$y_u=mx_u\ , \qquad y_v=nx_v \qquad (m=r'/r\ , \quad n=s'/s)\ , \tag{94}$$

so that we have

$$(y-mx)_u=-m_ux\ , \qquad (y-nx)_v=-n_vx\ . \tag{95}$$

Consequently the net N_x is conjugate to the congruence whose focal nets are generated by the points $y-mx$, $y-nx$. Similarly, the net N_y is shown to be conjugate to the same congruence by writing equations (94) in the form

$$[(y-mx)/m]_u=-m_uy/m^2\ , \qquad [(y-nx)/n]_v=-n_vy/n^2\ .$$

The reader may discuss the special case in which $r'=r$, $s'=s$.

It remains to determine *all congruences that are conjugate to a given net.* Let us consider a net N_x with x satisfying equation (1), and let us seek to determine the focal nets N_η, N_ζ of a congruence conjugate to N_x. Let N_η be the focal net that is harmonic to the congruence xx_1; then we can write

$$\eta_u=\psi x+\beta\eta\ , \qquad \eta_v=\varphi x_1+\delta\eta\ . \tag{96}$$

Equating the two expressions for η_{uv} derivable from these equations, and replacing η_u, η_v, x_{1u}, x_v by expressions equivalent to them, we obtain a linear equation in x, x_1, η, all of whose coefficients must vanish. Thus we find for equations (96) three integrability conditions, one of which is $\beta_v=\delta_u$. Consequently a transformation $\eta=\lambda\bar{\eta}$ with

$$\lambda_u-\beta\lambda=\lambda_v-\delta\lambda=0$$

can be made which will reduce both of β and δ to zero. Then equations (96) become

$$\eta_u=\psi x\ , \qquad \eta_v=\varphi x_1\ . \tag{97}$$

The two integrability conditions of these equations are

$$\psi=\varphi_u+b\varphi\,,\qquad \psi_v+a\psi=H\varphi\,. \tag{98}$$

Elimination of ψ shows that *the function* φ *is a solution of the adjoint of equation* (1), namely,

$$\varphi_{uv}=(c-a_u-b_v)\varphi-a\varphi_u-b\varphi_v\,. \tag{99}$$

Therefore *one focal net* N_η *of any congruence conjugate to the net* N_x *represented by equation* (1) *is determined by equations* (97) *wherein the function* φ *is a solution of equation* (99) *and* ψ *is determined by the first of* (98). *The other focal net* N_ζ *of the congruence is given by*

$$\zeta=\eta-\varphi x\,, \tag{100}$$

as is shown from the second of the following equations which are readily derivable from (100) in the presence of equations (97), (98):

$$\zeta_u=-\varphi x_{-1}\,,\qquad \zeta_v=-(\varphi_v+a\varphi)x\,. \tag{101}$$

When $\varphi=\psi=0$ the points η and ζ coincide in a fixed point and a congruence $\Gamma_{\eta\zeta}$ is a bundle of lines.

33. Polar sequences of Laplace. When a sequence of Laplace and a hyperquadric are given in a linear space of n dimensions S_n, there is determined another sequence in the space S_n by means of the polar relation between point and hyperplane with respect to the hyperquadric. The relation between the two sequences is a reciprocal one, and the sequences are called *polar sequences* with respect to the hyperquadric. Such sequences will now be studied analytically. The exposition will follow very closely that of Tzitzéica, but the fundamental ideas are due to Darboux.

Let us consider* an unlimited sequence of Laplace immersed in a space S_n and determined by a net N_x with x satisfying equation (1). Let us also consider in the space S_n a non-singular hyperquadric Q_{n-1}. The equation of Q_{n-1} can be written in the canonical form $\Sigma x^2=0$, the summation extending from 1 to $n+1$. The points $x_1, x_2, \ldots, x_n$, which generate nets of the Laplace sequence, determine a hyperplane S_{n-1}; the coordinates y of the pole of this hyperplane with respect to the hyperquadric Q_{n-1} satisfy the equations

$$\Sigma yx_1=0,\ \ldots\,,\qquad \Sigma yx_n=0\,, \tag{102}$$

* Tzitzéica, 1924. 3, p. 121.

and, if the proportionality factor of y is suitably chosen, also satisfy

$$\Sigma yx=1. \tag{103}$$

Similarly, the points $x, x_1, \ldots, x_{n-1}$ determine a hyperplane whose pole with respect to the hyperquadric Q_{n-1} has coordinates y_1 satisfying

$$\Sigma y_1x=0\ , \qquad \Sigma y_1x_1=0, \ldots , \qquad \Sigma y_1x_{n-1}=0\ , \tag{104}$$

and, with proper choice of proportionality factor, also satisfying

$$\Sigma y_1x_{-1}=1\ . \tag{105}$$

If we regard the space S_n as determined by the points $x, x_1, \ldots, x_n$, we can write

$$x_{-1}=dx+d_1x_1+\ldots+d_nx_n\ , \tag{106}$$

where the coefficients $d, d_1, \ldots, d_n$ are scalar functions of u, v.

We shall deduce some analytic consequences of the equations just written and use them to prove that *the point y describes a net N_y when u, v vary*, and that *the point y_1 also describes a net which, as the notation indicates, is the first Laplace transform of the net N_y.* Thus a y-sequence is defined which is called *the polar sequence* of the original x-sequence with respect to the hyperquadric Q_{n-1}. For the purpose of the proof it is sufficient to show that there exist four scalar functions p, q, r, s of u, v such that

$$y_v=py+qy_1\ , \qquad y_{1\,u}=ry+sy_1\ . \tag{107}$$

In preparation for the existence proof we observe that equations (102), (103), (106) imply

$$\Sigma yx_{-1}=d\ . \tag{108}$$

Differentiation of equation (103) and use of equations (2), (102), (103) give

$$\Sigma y_vx=-a\ . \tag{109}$$

Differentiation of equation (108) and substitution for $x_{-1\,v}$ lead to

$$\Sigma y_vx_{-1}=d_v-ad-K\ . \tag{110}$$

Finally, differentiation of $\Sigma y x_r = 0 (r = 1, \ldots, n-1)$ and substitution for $x_{r\,v}$ from the first of equations (22) yield

$$\Sigma y_v x_r = 0 \qquad (r = 1, \ldots, n-1) . \tag{111}$$

How the $n+1$ equations

$$\Sigma(y_v - py - qy_1)x_{-1} = 0 , \qquad \Sigma(y_v - py - qy_1)x = 0, \ldots ,$$
$$\Sigma(y_v - py - qy_1)x_{n-1} = 0$$

in the $n+1$ unknown values of the expression within parentheses are such that the determinant of their coefficients is not zero, since the vanishing of this determinant would imply the linear dependence of the points $x_{-1}, x, \ldots, x_{n-1}$. Hence there exist two functions p, q of u, v, such that the first of equations (107) is satisfied. In fact, it is easy to verify by means of the equations satisfied by y_v, y, y_1 that the actual formulas for p, q are

$$p = -a , \qquad q = d_v - K . \tag{112}$$

The demonstration for the second of equations (107) can be made similarly.

Geometrically, the net N_x which was used to determine the x-sequence was any net whatever of the sequence. Therefore we can generalize the result just obtained and say that *the pole y_r of the hyperplane determined by the points $x_{-r+1}, x_{-r+2}, \ldots, x_{-r+n}$ is the* rth *transform of the net N_y in the positive direction if $r > 0$, and in the negative direction if $r < 0$.*

The foregoing observation permits us to demonstrate that *the polar relation between the two sequences is reciprocal,* by showing that *the x-sequence is the polar of the y-sequence.* If we look at equations (104), which are satisfied by y_1, and write the analogous equations which are satisfied by each of y_2, $y_3, \ldots, y_n$, namely,

$$\left\{\begin{array}{lll} \Sigma y_2 x_{-1} = 0 , & \Sigma y_2 x = 0 , \ldots , & \Sigma y_2 x_{n-2} = 0 , \\ \quad\vdots & \quad\vdots & \quad\vdots \\ \Sigma y_n x_{-n+1} = 0 , & \Sigma y_n x_{-n+2} = 0 , \ldots , & \Sigma y_n x = 0 , \end{array}\right. \tag{113}$$

we observe that among these equations appear

$$\Sigma x y_1 = 0 \qquad \Sigma x y_2 = 0, \ldots , \qquad \Sigma x y_n = 0 . \tag{114}$$

Therefore *the point x is the pole of the hyperplane determined by the points* $y_1, \ldots, y_n$, just as the point y is the pole of the hyperplane determined by the points $x_1, \ldots, x_n$; and the argument previously made can be applied, with x and y interchanged, to establish the desired result.

EXERCISES

1. If $H=0$, then $x_1 = Ve^{\int b\,du}$ and consequently the general solution of equation (1) is

$$x = e^{\int a\,dv}\left[\int Ve^{\int b\,du - \int a\,dv}dv + U\right],$$

where U and V are arbitrary functions of u alone and v alone respectively. In general if $H_r K_{-s}=0$, where r, s are any positive integers or zero, equation (1) can be integrated by quadratures.

2. If x satisfies equation (1) and also $x_{vv}=qx+\delta x_v$, calculate the integrability conditions for this system of two equations, and hence show that the locus of the point x is a developable surface with its v-curves for generators, these lines being tangents of a u-curve into which the net N_1 degenerates.

3. The x in the parametric vector equation (II, 11) of any developable surface satisfies two equations of the form (10), namely, $x_{uu}=0$, $x_{ut}+ux_u-x=0$. Conversely if a surface S whose parametric vector equation is $x=x(u, v)$ in space S_n $(n>3)$ is such that x satisfies two essentially distinct equations of the form (10), then S is developable. Every surface in space S_3 is an integral surface of two such equations.
SEGRE, 1907. 2, p. 1054

4. On a developable surface in space S_n the generators and any one-parameter family of curves form a conjugate net.

5. If two Laplace equations $L(x)=0$ and $\bar{L}(\bar{x})=0$ have their corresponding invariants equal, so that $H=\bar{H}$, $K=\bar{K}$, then there exists a transformation $x=\lambda\bar{x}$ that will convert one of these equations into the other.

6. Use Green's characterization of plane nets with equal invariants, together with the facts that when the Γ_1-curves on a surface S in space S_3 are indeterminate the reciprocal congruence Γ_2 is harmonic to S, and that the u-tangent at each point on the surface S_σ intersects the corresponding line l_1 as does also the corresponding v-tangent on the surface S_ρ, to demonstrate synthetically the theorem of Koenigs on plane nets with equal invariants. GREEN, 1919. 1, p. 106

7. Study the effect of the transformation

$$x=\lambda\bar{x}\,, \qquad \bar{u}=U(u)\,, \qquad \bar{v}=V(v) \qquad\qquad (\lambda U'V'\neq 0)$$

on equations (26). In particular, by means of a transformation $x=\lambda\bar{x}$, reduce these equations to the canonical form for which $\alpha+b=0$, $\delta+a=0$. Determine the most general transformation preserving this canonical form.
WILCZYNSKI, 1911. 2, p. 477

8. Prove that the differential equation of *the associate conjugate net* of the parametric conjugate net on a surface in space S_3 is $Ldu^2 - Ndv^2 = 0$, using the definition that the associate tangents at each point P_x separate the parametric tangents harmonically. Prove also that the differential equation of *the anti-ray net* of the parametric net is

$$LHdu^2 - 2nLNdudv - NKdv^2 = 0 \ ,$$

using the definition that the anti-ray tangents at P_x are the harmonic reflections of the ray tangents in the parametric tangents. The associate tangents, the anti-ray tangents, and the axis tangents belong to the same involution in case $n(\log r)_{uv} = 0$. Then the parametric net is either harmonic, or isothermally conjugate, or both.

GREEN, 1916. 1, p. 313

9. (A J net is a conjugate net in space S_3 which is isothermally conjugate and has equal point invariants.) If a surface has on it a J net, then, by choosing the asymptotic parameters of Chapter III so that the J net has the equation $dv^2 - du^2 = 0$, show that $\beta_u = \gamma_v$. For a J net the ray and associate ray intersect on the second edge e_2 of Green (see Ex. 26 of Chap. III).

FUBINI and ČECH, 1926. 1, p. 106; ČECH, 1929. 4, p. 1333

10. A focal plane of the axis at a point P_x of a conjugate net in space S_3 is tangent (at the corresponding focal point) to both ruled surfaces of axes that contain the curves of the net through P_x. A variable plane through the axis touches these two ruled surfaces in a pair of points in a projectivity which has the focal points of the axis for double points. For the parametric net associated with system (31) the invariant of this projectivity is the ratio of the roots of equation (35).

11. The axis curves (35) of a net N_x are indeterminate in case $\mathscr{H} = n = \mathscr{K} = 0$. If not indeterminate they coincide with the fundamental net N_x in case $\mathscr{H} = \mathscr{K} = 0$. In both cases N_x consists of plane curves, so that the nets N_1 and N_{-1} are on developables; in the first case the nets N_1 and N_{-1} are on cones with a common vertex, through which all of the axes of the net N_x pass. Dually, the ray curves (48) are indeterminate in case $H = n = K = 0$. If not indeterminate they coincide with N_x in case $H = K = 0$. In both cases N_x consists of cone curves, so that N_1 and N_{-1} reduce to curves; in the first case these curves are plane curves in the same plane, in which all the rays of the net N_x lie.

12. If the rays of a net in space S_n form a congruence in the sense of Section 28, then either $H = K = 0$, the number of dimensions n being unrestricted, or else $n = 2$ or $n = 3$.

13. Eliminating y from equations (31), calculate the coefficients of the equations used by Green for the study of a conjugate net, which can be written in the form

$$x_{uu} = ax_{vv} + bx_u + cx_v + dx \ ,$$
$$x_{uv} = b'x_u + c'x_v + d'x \ .$$

Hence prove that the invariants $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{D}$ defined in Green's notation by

$$8\mathfrak{B}'=4b'+2c/a-(\log a)_v\ ,\qquad 8\mathfrak{C}'=4c'-2b+(\log a)_u\ ,$$
$$\mathfrak{D}=d+ab'^2-c'^2+ab'_v-c'_u+b'c+bc'\ ,$$

are given in the notation of system (31) by

$$8\mathfrak{B}'=4a-2\delta+(\log r)_v\ ,\qquad 8\mathfrak{C}'=4b-2\alpha-(\log r)_u\ ,$$
$$\mathfrak{D}=-2nL\ .$$

GREEN, 1915. 3 and 1916. 1

14. In conjugate parameters the differential equation of the curves of Darboux (see Ex. 13) is

$$\mathfrak{C}'du^3-3\mathfrak{B}'du^2dv-3r\mathfrak{C}'dudv^2+r\mathfrak{B}'dv^3=0\ ,$$

and the equation of the curves of Segre is

$$\mathfrak{B}'du^3+3r\mathfrak{C}'du^2dv-3r\mathfrak{B}'dudv^2-r^2\mathfrak{C}'dv^3=0\ .$$

If the parametric conjugate net of the present chapter is the net $dv^2-\lambda^2du^2=0$ in the asymptotic parameters of Chapter III, prove that the equations $\mathfrak{B}'=0$ and $\mathfrak{C}'=0$ are respectively equivalent to the equations $\beta du^3+\gamma dv^3=0$ and $\beta du^3-\gamma dv^3=0$ in asymptotic parameters. LANE, 1922. 3, p. 287, and 1926. 4, p. 163

15. If a surface has on it an R net, then, by choosing the asymptotic parameters of Chapter III so that the R net has the equation $dv^2-du^2=0$, show that $\beta_v=\gamma_u$ (see Ex. 22 of Chap. V).

16. Using the fact that a conjugate net is *harmonic* in case the foci of each ray separate the ray-points thereon harmonically, prove that the net $dv^2-\lambda^2du^2=0$ in asymptotic parameters is harmonic in case $F+G\lambda^2=0$, the definitions of F, G being given in equations (III, 37) in which a, b are given by equations (III, 68).

17. Two congruences harmonic to one net are conjugate to another net (called a *derived net* of the first). If the congruences are generated by the line $\rho\sigma$ given by equations (81), (82) and the line ρ', σ' given by the same equations with θ' in place of θ, the derived net is generated by the point X defined by

$$X=\begin{vmatrix} x & x_u & x_v \\ \theta & \theta_u & \theta_v \\ \theta' & \theta'_u & \theta'_v \end{vmatrix}\ .$$

18. Two congruences conjugate to one net are harmonic to another net (called a *derivant net* of the first). If the congruences are generated by the line $\eta\zeta$ given by

equations (97), (100) and the line η', ζ' given by the same equations with φ' in place of φ, the derivant net is generated by the point Y defined by

$$Y = \varphi'\eta - \varphi\eta' = \varphi'\zeta - \varphi\zeta' .$$

EISENHART, 1923. 3, p. 26

19. If ξ is defined by equation (50) and η by

$$\eta = e^{-\theta}(x_{-1}, y, x_1) ,$$

then ξ, η satisfy a system of equations of the form (31), of which the second is equation (51) and of which the first and third are

$$\xi_{uu} = p'\xi + \alpha'\xi_u + L\eta ,$$
$$\xi_{vv} = q'\xi + \delta'\xi_v + N\eta ,$$

where

$$p' = A[b - \alpha + (\log L/A)_u] - nL ; \qquad \alpha' = -\alpha + (\log LN)_u ,$$
$$q' = B[a - \delta + (\log N/B)_v] + nN , \qquad \delta' = -\delta + (\log LN)_v .$$

20. The projection of a conjugate net from a point onto a hyperplane is a conjugate net, and the projection of a congruence from a point onto a hyperplane is a congruence.

21. If a conjugate net is immersed in space S_n, and if a congruence is in a hyperplane S_{n-1} and is such that there is just one of its lines in each tangent plane of the net, then the congruence is harmonic to the net.

22. Carry out an investigation analogous to that indicated in Exercise 31 of Chapter III, starting with the parametric conjugate net of equations (31) instead of with the asymptotic net. Consider, in particular, a bundle of nets on an integral surface S of equations (31), consisting of all nets every one of which has the property that at every point P_x of S its tangents separate the parametric conjugate tangents *harmonically*, and consider a pencil of nets in this bundle, determined by an arbitrary net of the bundle. Prove that the rays of a point P_x with respect to all the nets of this pencil form a flat pencil of lines. Find the coordinates of the center of this pencil. LANE, 1926. 4, p. 164

23. Use equation (71) with λ, μ unrestricted and with η, ζ satisfying equations (69) to prove that an unspecialized transversal surface of an arbitrary congruence in space S_n is an integral surface of a pair of differential equations of the form

$$x_{uuv} = ax_{uu} + hx_{uv} + lx_u + mx_v + dx ,$$
$$x_{uvv} = h'x_{uv} + b'x_{vv} + l'x_u + m'x_v + d'x .$$

P. TZITZÉICA, 1927. 2, p. 582

24. Every integral surface of a completely integrable system of equations of the form appearing in Exercise 23 is a transversal surface of a congruence.

B. SEGRE, 1927. 12

25. Calculate the effect of the transformation (59) on the coefficients of system (31). Prove that when $n \neq 0$, it is possible to choose μ so that after the transformation one has $n=1$. Show that it is always possible to reduce system (31) to a canonical form for which

$$b+\alpha+A=0\ , \qquad a+\delta+B=0$$

26. In the notation of equations (67) prove that

$$\mathscr{H}_1=\mathscr{K}\,, \qquad \mathscr{K}_1=2\mathscr{K}-\mathscr{H}-(\log \mathscr{K})_{uv}\ ,$$

$$8\mathfrak{B}_1'=8\mathfrak{B}'+(\log H^3\mathscr{K}/r^2)_v\ , \qquad 8\mathfrak{C}_1'=8\mathfrak{C}'-(\log r^2H\mathscr{K}^3)_v\ ,$$

$$2H\mathscr{K}n_1=2HLn-\{(H/r)[4\mathfrak{B}'+(\log H/r^{1/2})_v]\}_v\ .$$

27. If a sequence of Laplace is periodic of period k so that the net N_k coincides with the net N_x, and k is numerically the smallest integer for which this happens, then the sequence is in a linear space of $k-1$ dimensions.

28. In ordinary space the u-curves of a conjugate net are plane curves in case $\mathscr{H}=0$ and the v-curves are plane in case $\mathscr{K}=0$.

29. Three consecutive u-tangents constructed at a point x and two consecutive points of a curve C_λ on an integral surface of system (31) determine a quadric Q_u whose equation referred to the tetrahedron x, x_{-1}, x_1, y is

$$\begin{aligned}(L-N\lambda^2)x_3^2-\lambda[4(\lambda\mathfrak{B}'+4\mathfrak{C}')-(\log \lambda r^{1/2})']x_3x_4\\ +\lambda(2n\lambda-\lambda^2K/L+\mathscr{H}/N)x_4^2+2\lambda(\lambda x_1x_4-Lx_2x_3)=0\ .\end{aligned}$$

Write the equation of the quadric Q_v and study the relations of these quadrics.

CHAPTER V

TRANSFORMATIONS OF SURFACES

Introduction. A transformation between two geometrical configurations is a one-to-one correspondence between their generating elements. From this point of view a transformation between two surfaces regarded as point loci is a one-to-one correspondence between their generating points. We may speak of a point transformation between two nets of curves on two surfaces, meaning by this locution a point transformation between the surfaces sustaining the nets. Some transformations of this kind have already occurred in this book. One instance in Section 28 is the transformation of Laplace between the focal nets of a congruence, two corresponding points being the focal points of a generator of the congruence. Another instance in Section 32 is the transformation of Levy between a focal net of a congruence and any net conjugate to the congruence.

In this chapter several other transformations of surfaces will be discussed. The first of these is *the fundamental transformation, or transformation F*, of Jonas and Eisenhart, which is the transformation between two conjugate nets that are conjugate to the same congruence, two corresponding points being on the same generator of the congruence. *The transformation of Koenigs* to be considered in Section 35, and *the transformation of Ribaucour* in Section 39, are special cases of this transformation. In Section 36 the general analytic point transformation between two analytic surfaces is introduced, and the configuration composed of two surfaces in the relation of such a transformation and having the additional property that their tangent planes at corresponding points intersect in straight lines is briefly studied. These considerations are further specialized in Section 37 by restricting both surfaces to be in the same ordinary space S_3. In Section 38 the emphasis is on the correspondence itself rather than on the configuration composed of the two surfaces related by the correspondence, and in this section the two surfaces are supposed to be in two ordinary spaces S_3, not necessarily the same, one of the surfaces lying in one of the spaces and one in the other.

It is well known that *ordinary ruled space*, that is, ordinary space regarded as made up of its straight lines, is four-dimensional. Section 40 is devoted to an account of a correspondence between the lines of this space and the points of a hyperquadric in space S_5. In the next section this transformation is applied to the theory of surfaces in ordinary ruled space.

In Section 42 Fubini's theory of W congruences is summarized to prepare the way for the study in the final section of Terracini's theory of W congruences in ordinary ruled space.

34. The fundamental transformation. We begin with some definitions. *The fundamental transformation, or transformation F, between two surfaces in space S_n is a one-to-one correspondence between their points such that the lines joining corresponding points form a congruence whose developables intersect both surfaces in conjugate nets which are conjugate to the congruence.* The surfaces cannot be focal surfaces of the congruence. It is possible also to say that the correspondence thus established between two conjugate nets by means of a common conjugate congruence is a transformation F between the nets themselves as well as between the surfaces sustaining them. The nets are called *F transforms* of each other, and are sometimes said to be *in relation F*. It was shown in Section 32 that if two conjugate nets are conjugate to one congruence they are harmonic to another congruence, and conversely; the first congruence will be called *the conjugate congruence,* and the second *the harmonic congruence,* of the transformation F involved. If the conjugate congruence is a bundle of lines the two nets are said to be *radially related,* and if the harmonic congruence is in a hyperplane the two nets are said to be *perspective.*

Several special cases of the transformation F have been considered from time to time by various geometers. The general transformation seems to have been studied* first by Jonas and Eisenhart. The reader who wishes to see a more complete treatment of this transformation than that offered here should consult the book† by Eisenhart bearing the same title as this chapter. For instance, Eisenhart devotes considerable attention to the so-called *theorems of permutability* which are scarcely more than mentioned here. The point of view adopted by Eisenhart is predominantly that of metric geometry.

In this section all nets that are F transforms of a given net are determined analytically by very simple formulas. Some of the geometrical relations of a net and an F transform of it are deduced. Two cross ratios called *the conjugate invariant* and *the harmonic invariant* of the transformation are defined. The section closes with a glance at the completely integrable system of partial differential equations that define, except for a projective transformation, a pair of surfaces in relation F in ordinary space.

The first problem is *to determine analytically all nets that are F transforms of a given net.* For this purpose let us consider in space S_n a net N_x gener-

* Jonas, 1915. 4; Eisenhart, 1917. 3; 1923. 3, pp. 34 f.

† Eisenhart, 1923. 3.

ated by a point P_x whose coordinates x are functions of two independent variables u, v and satisfy an equation of Laplace,

$$x_{uv}=cx+ax_u+bx_v \ . \tag{1}$$

Equations (IV, 81), (IV, 82) show that any congruence $\rho\sigma$ harmonic to the net N_x consists of lines crossing the tangents of N_x in the points ρ, σ defined by

$$\rho=(x/\theta)_u \ , \qquad \sigma=(x/\theta)_v \ , \tag{2}$$

where θ is a solution of equation (1). The same result can also be reached by observing that equation (1) can be written in the form

$$(x/\theta)_{uv}=A(x/\theta)_u+B(x/\theta)_v \ , \tag{3}$$

wherein A, B are defined by

$$A=a-(\log\theta)_v \ , \qquad B=b-(\log\theta)_u \ . \tag{4}$$

If a second net N_y is also harmonic to the congruence $\rho\sigma$, the proportionality factor of y can be chosen so that each of the lines $\rho\sigma$ crosses the corresponding tangents of N_y in the points y_u, y_v. Since the points ρ, y_u coincide at one focal point of the line $\rho\sigma$, and the points σ, y_v at the other, it follows that

$$y_u=m(x/\theta)_u \ , \qquad y_v=n(x/\theta)_v \ , \tag{5}$$

where m, n are scalar functions of u, v which are restricted to satisfy the integrability conditions that result from equating the two expressions for y_{uv} derivable from equations (5) in the presence of equation (3), namely,

$$m_v=A(n-m) \ , \qquad n_u=B(m-n) \ . \tag{6}$$

The point equation of the net N_y is $(y_u/m)_v=(y_v/n)_u$, and can be reduced to

$$y_{uv}=Any_u/m+Bmy_v/n \ . \tag{7}$$

The solution of the problem at hand can now be formulated in the following theorem:

When a net N_x with the point equation (1) *is given, any net N_y which is an F transform of N_x can be determined by first finding a solution θ of equation* (1), *by next calculating A, B in* (4), *by then solving equations* (6) *for m, n, and by finally performing the quadrature indicated by equations* (5) *for y.*

A more general analytic representation of a transformation F than the one just explained results if no restriction on the proportionality factor of y is made. In order to write the equations analogous in this case to equations

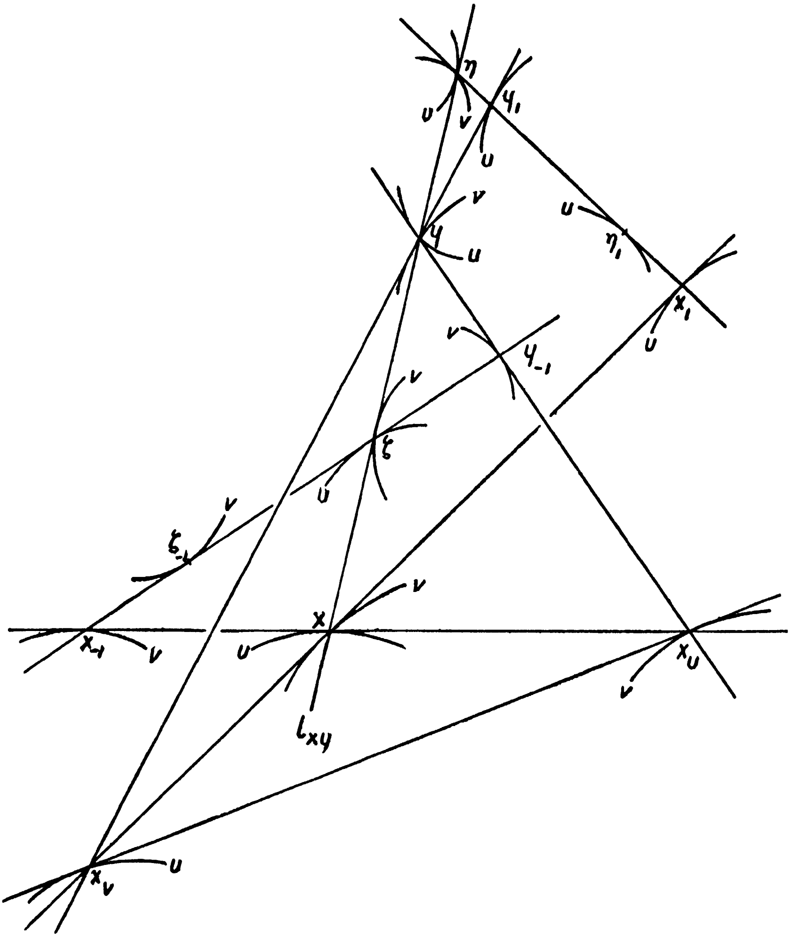

FIG. 26

(1), . . . , (7), it is sufficient to replace y wherever it appears in the equations of the preceding paragraph by y/φ, where φ is a solution of the point equation of the net N_y. On the other hand, a simpler analytic representation than either of these can be obtained by supposing that the proportionality factor

of x has been chosen so that the harmonic congruence of the transformation consists of lines crossing the tangents of the net N_x in the points x_u, x_v. This simplification amounts to taking $\theta=1$, and hence $c=0$, in the preceding paragraph. The formulas employed in this analytic representation of a transformation F, which is in a sense *the simplest analytic representation possible,* are written together below for convenience in the order in which they are used in passing from the net N_x to the net N_y:

$$\tag{8} \begin{cases} x_{uv}=ax_u+bx_v \ , \\ m_v=a(n-m) \ , \qquad n_u=b(m-n) \ , \\ y_u=mx_u \ , \qquad y_v=nx_v \ , \\ y_{uv}=any_u/m+bmy_v/n \ . \end{cases}$$

Equations (8) will now be employed to deduce some geometrical relations among the various parts of the configuration composed of two nets in relation F together with the associated conjugate and harmonic congruences (see Fig. 26). That the focal points of a line l_{xy} are the points P_η, P_ζ defined by

$$\tag{9} \eta=y-mx \ , \qquad \zeta=y-nx$$

is readily verified after observing that differentiation gives

$$\tag{10} \begin{cases} \eta_u=-m_u x \ , & \eta_v=(n-m)x_1 \ , \\ \zeta_u=(m-n)x_{-1} \ , & \zeta_v=-n_v x \ , \end{cases}$$

the Laplace transformed points x_1, x_{-1} of the point P_x being defined by the usual formulas,

$$\tag{11} x_1=x_v-ax \ , \qquad x_{-1}=x_u-bx \ .$$

The Laplace transformed points y_1, y_{-1} of the point P_y are given by

$$\tag{12} y_1=n(x_1-a\eta/m) \ , \qquad y_{-1}=m(x_{-1}-b\zeta/n) \ .$$

The point equations of the nets N_η, N_ζ are respectively

$$\tag{13} \begin{cases} \eta_{uv}=(m-n)H\eta_u/m_u+m_u\eta_v/(m-n) \ , \\ \zeta_{uv}=n_v\zeta_u/(n-m)+(n-m)K\zeta_v/n_v \ , \end{cases}$$

where H, K are the Laplace-Darboux invariants of the net N_x which were defined by formulas (IV, 7). Hence the Laplace transforms η_1, η_{-1}, ζ_1, ζ_{-1} of the points P_η, P_ζ are given by

$$\left\{\begin{array}{ll} m_u\eta_1=(n-m)(H\eta+m_u x_1)\ , & (n-m)\eta_{-1}=m_u\zeta\ , \\ (m-n)\zeta_1=n_v\eta\ , & n_v\zeta_{-1}=(m-n)(K\zeta+n_v x_{-1})\ . \end{array}\right. \tag{14}$$

Therefore *the points* x_1, y_1, η, η_1 *are collinear, as are also the points* x_{-1}, y_{-1}, ζ, ζ_{-1}. The first focal plane of the line l_{xy}, which is tangent to the net N_η at the point P_η, contains the points x_1, y_1, η_1 in addition to the points x, y, η, ζ; the second focal plane contains the points x_{-1}, y_{-1}, ζ_{-1}.

There are two cross ratios of particular interest associated with a fundamental transformation. The first of these is the cross ratio of two corresponding points x, y and the two focal points η, ζ on the line xy. This is denoted by R and is called *the conjugate invariant* of the transformation. By means of equations (9) we find

$$R=n/m\ . \tag{15}$$

As an application let us consider the special case $R=1$. In this case $m=n$ and equations (9) show that the focal points η, ζ coincide. Moreover, the second and third of equations (8) show that $m=n=\text{const.}$ Equations (10) show that the point η is fixed. Therefore the conjugate congruence xy of the transformation is a bundle of lines, and the nets N_x, N_y are radially related; the point equations of these nets, namely, the first and last of (8) are identical. The converse reasoning may be supplied by the reader. The special case $R=-1$ will be discussed in the next section.

The second cross ratio referred to above is that of the tangent planes of the nets N_x, N_y at two corresponding points x, y and the first and second focal planes of the corresponding generator of the harmonic congruence, the first focal plane being tangent at the point x_v to the locus of this point and the second focal plane bearing the same relation to the point x_u. This cross ratio is denoted by S and is called *the harmonic invariant* of the transformation.

If the configuration composed of two nets N_x, N_y *in relation* F *is in ordinary space* S_3, *then the four coordinates* x *and the four coordinates* y *form four pairs of solutions of a completely integrable system of differential equations which can be reduced* to the form*

$$\left\{\begin{array}{l} x_{uu}=px+\alpha x_u+\beta x_v+Ly\ , \\ x_{uv}=\qquad ax_u+bx_v\ , \\ x_{vv}=qx+\gamma x_u+\delta x_v+Ny\ , \\ y_u=mx_u\ , \qquad y_v=nx_v\ . \end{array}\right. \tag{16}$$

* Lane, 1929. 1, p. 460.

The reduction does not differ essentially from that employed in obtaining equations (8), except that one observes that the space S_3 may be regarded as determined by the points x, x_u, x_v, y, and that the points x_{uu}, x_{vv} are certainly linearly dependent on these points. The coefficients of equations (16) satisfy the following *integrability conditions*, which are found by the usual method from the equations

$$(x_{uu})_v=(x_{uv})_u\ ,\qquad (x_{uv})_v=(x_{vv})_u\ ,\qquad (y_u)_v=(y_v)_u$$

and the fact that the points x, x_u, x_v, y are not coplanar:

$$\tag{17}\begin{cases} a_u+ab=a_v+\beta\gamma\ , & b_v+ab=\delta_u+\beta\gamma\ , \\ b_u+b^2+a\beta=\beta_v+ba+nL+\beta\delta+p\ , & \\ a_v+a^2+b\gamma=\gamma_u+a\delta+mN+a\gamma+q\ , & \\ p_v=ap-q\beta\ , & q_u=bq-p\gamma\ , \\ L_v=aL-\beta N\ , & N_u=bN-\gamma L\ , \\ m_v=a(n-m)\ , & n_u=b(m-n)\ . \end{cases}$$

Not only the original pair of nets but also any pair into which this pair can be projected gives rise to the same system (16). Conversely, *equations* (16) *with the integrability conditions* (17) *satisfied define a pair of nets in relation* F *in space* S_3, *except for a projective transformation.*

The conjugate invariant R of a pair of nets as just defined in space S_3 is still given by (15); the harmonic invariant S can be calculated by observing that the first and second focal planes of a line x_vx_u of the harmonic congruence intersect the corresponding line xy in the points $qx+Ny$, $px+Ly$ respectively. The result of the calculation is

$$\tag{18} S=pN/qL\ .$$

Additional results on fundamental transformations in ordinary space may be found in Exercises 3, 4, 25.

35. The conics and transformation of Koenigs. Conjugate nets with equal Laplace-Darboux invariants H, K constitute a class of nets of considerable interest. Plane nets with equal invariants were characterized geometrically in Section 29, and conjugate nets with equal invariants in ordinary space were characterized in Section 31. We shall give in the present section a geometric characterization of conjugate nets with equal invariants in space S_n, using some conics called *the conics of Koenigs*. Moreover, in this section we shall also consider the special kind of fundamental transformation called *the transformation of Koenigs* of one conjugate net

with equal invariants into another net of the same kind, and shall study briefly certain pencils of conics containing the conics of Koenigs at corresponding points of the two nets.

In order to define *the conics of Koenigs* we make the following observations. In the tangent plane at a point P_x of a net N_x in space S_n, with x satisfying equation (1), there is a pencil of conics tangent to the curve C_u at the point x_1 and tangent to the curve C_v at the point x_{-1}. In order to

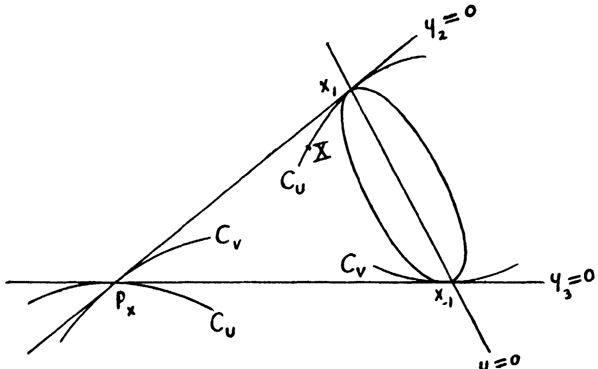

FIG. 27

write the equation of a general conic of this pencil, let us choose the points x, x_{-1}, x_1 as the vertices of a local triangle of reference in the tangent plane, with a unit point chosen so that a point $y_1x+y_2x_{-1}+y_3x_1$ may have local coordinates proportional to y_1, y_2, y_3. Then the equation of a general conic belonging to the pencil of conics tangent to the line $y_2=0$ at the point $(0, 0, 1)$, and tangent to the line $y_3=0$ at the point $(0, 1, 0)$ is of the form (see Fig. 27)

$$y_1^2-fy_2y_3=0 \qquad (f \text{ arbitrary}) . \tag{19}$$

It will be shown in the next paragraph that there is one conic of this pencil which has three-point contact with the curve C_u at the point x_1, and similarly one conic of the pencil which has three-point contact with the curve C_v at the point x_{-1}. These two conics are by definition *the conics of Koenigs* at the point P_x of the net N_x.

The equations of the conics of Koenigs can be derived in the following way. The coordinates of any point X near the point x_1 on the curve C_u that passes through the point x_1 can be represented by Taylor's expansion

as power series in the increment Δu corresponding to displacement from the point x_1 to the point X along C_u:

$$X = x_1 + x_{1\,u}\Delta u + x_{1\,uu}\Delta u^2/2 + \cdots .$$

Making use of equation (IV, 18) and the equation obtained therefrom by differentiating once with respect to u, we find

$$X = y_1 x + y_2 x_{-1} + y_3 x_1 ,$$

where, to terms of as high degree as will be needed,

$$(20) \qquad y_1 = H\Delta u + \cdots , \qquad y_2 = H\Delta u^2/2 + \cdots , \qquad y_3 = 1 + \cdots .$$

Demanding that equation (19) be satisfied by the power series (20) for y_1, y_2, y_3 identically in Δu as far as the terms of the second degree, we find $f = 2H$. If in place of C_u and x_1 we had used C_v and x_{-1} we should have found $f = 2K$. Thus we obtain *the equations of the conics of Koenigs,*

$$(21) \qquad y_1^2 - 2Hy_2y_3 = 0 , \qquad y_1^2 - 2Ky_2y_3 = 0 .$$

These conics coincide in case $H = K$. Thus the following theorem is established.

Conjugate nets with equal invariants in space S_n are characterized by the property that in the tangent plane at each point x of any one of them there exists a conic having three-point contact with the u-curve at the ray-point x_1 and also having three-point contact with the v-curve at the ray-point x_{-1}.

The definition of the transformation of Koenigs can be formulated in the following words. *Two conjugate nets N_x, N_y in relation F are said to correspond by a transformation of Koenigs in case the foci P_η, P_ζ of each line xy separate harmonically the points P_x, P_y thereon.* Then $R = -1$, and equation (15) shows that $n = -m$; consequently the second and third of equations (8) give

$$(22) \qquad m_v = -2am , \qquad m_u = -2bm .$$

It follows that $a_u = b_v$, and the net N_x has equal invariants. Moreover, in this case the first and last of equations (8) show that each coefficient of the point equation of the net N_y differs only in sign from the corresponding coefficient of the point equation of the net N_x; hence N_y also has equal invariants. So we reach the conclusion:

Each of two conjugate nets that correspond by a transformation of Koenigs has equal invariants.

We proceed to consider certain pencils of conics associated with the transformation of Koenigs. The tangent planes at corresponding points of two conjugate nets N_x, N_y in relation F in space S_n determine a space S_3 in which the points x, x_u, x_v, y can be used as the vertices of a local tetrahedron of reference with a unit point chosen so that a point $z_1x+z_2x_u+z_3x_v+z_4y$ may have local coordinates proportional to $z_1, \ldots, z_4$. In this coordinate system the conic (19) has the equations

$$z_4=(z_1+bz_2+az_3)^2-fz_2z_3=0 \, , \tag{23}$$

since the following relations hold between the coordinates of a point in the tangent plane of the net N_x in the two systems:

$$z_1=y_1-by_2=ay_3 \, , \qquad z_2=y_2 \, , \qquad z_3=y_3 \, , \qquad z_4=0 \, .$$

If a point in the tangent plane of the net N_y has coordinates y_1', y_2', y_3' referred similarly to the triangle y, y_{-1}, y_1, the following relations exist between these coordinates and the coordinates $z_1, \ldots, z_4$ of the same point:

$$z_1=0 \, , \qquad z_2=my_2' \, , \qquad z_3=ny_3' \, , \qquad z_4=y_1'-bmy_2'/n-any_3'/m \, .$$

Therefore a general conic of the pencil in the tangent plane of the net N_y, tangent to the curve C_u at the point y_1 and to C_v at the point y_{-1}, has the equations

$$z_1=(mnz_4+bmz_2+anz_3)^2-gmnz_2z_3=0 \qquad (g \text{ arbitrary}) \, . \tag{24}$$

These two pencils of conics determine involutions on the line $z_1=z_4=0$, the double points of the first involution being obtained by solving

$$b^2z_2^2-a^2z_3^2=0 \, ,$$

and those of the second by solving

$$b^2m^2z_2^2-a^2n^2z_3^2=0 \, .$$

These two involutions are the same in case $m^2=n^2$, that is, in case the transformation F is either a radial transformation or else a transformation of Koenigs. In the latter case two conics that intersect the line $z_1=z_4=0$ in the same pair of points have their parameters connected by the relation

$$f+g=4ab \, . \tag{25}$$

Two such conics determine a pencil of quadric surfaces,

$$\text{(26)} \quad \begin{cases} z_1^2+b^2z_2^2+a^2z_3^2+m^2z_4^2+2bz_1z_2+2az_1z_3+2Az_1z_4+(2ab-f)z_2z_3 \\ \qquad\qquad\qquad\qquad\qquad\qquad -2bmz_2z_4+2amz_3z_4=0 \ , \end{cases}$$

the parameter of the pencil being A. If $f(f-4ab)\neq 0$, so that the two conics are proper conics, there are two cones besides the planes $z_1z_4=0$ in the pencil, and for them $A=\pm m$ (see Ex. 9).

36. Pairs of surfaces with their points in one-to-one correspondence. The configuration composed of two surfaces with their points in one-to-one correspondence is of frequent occurrence in differential geometry. Instances of this are two surfaces sustaining conjugate nets in the relation of a Laplace transformation, or a Levy transformation, or a transformation F. It is the purpose of this section to consider, in the first place, *the general analytic point transformation between two analytic surfaces.* Afterward the restriction is imposed that the tangent planes at corresponding points of the two surfaces intersect in straight lines. This restriction may, from one point of view, be considered as mild, since all pairs of surfaces with their points in correspondence in ordinary space are of necessity subjected to it, but from another point of view it is quite severe, since two planes in space $S_n(n>4)$ ordinarily do not intersect at all.

We wish to write *the equations of the general analytic point transformation between two analytic surfaces.* In space S_n let us consider two proper analytic surfaces S_x, S_y with the respective parametric vector equations

$$x=x(u, v) \ , \qquad y=y(\xi, \eta) \ .$$

On the surface S_x let us consider a region in which points P_x are in one-to-one correspondence with pairs of values of u, v, and similarly for S_y. Then a one-to-one correspondence between points P_x, P_y is equivalent to a functional relation between the parameters u, v and ξ, η which may be expressed by equations of the form

$$\text{(27)} \qquad \xi=\xi(u, v) \ , \qquad \eta=\eta(u, v) \qquad (\xi_u\eta_v-\xi_v\eta_u\neq 0) \ ,$$

the functions ξ, η being supposed analytic. These are the desired equations of transformation.

A curve on either one of the surfaces S_x, S_y will be said to correspond to a curve on the other in case the two curves are generated by corresponding points. All the curves through a point P_x on the surface S_x have at P_x tangent lines which form a pencil in the tangent plane of S_x and with its

center at P_x; these curves correspond to curves through a point P_y on the surface S_y whose tangents at P_y form a pencil in the tangent plane of S_y and with its center at P_y. So there is established a line correspondence between the pencils of tangents at corresponding points P_x, P_y, corresponding lines being tangents of corresponding curves. *This line correspondence will now be shown to be a projectivity.* The direction dv/du of a curve at the point P_x and the direction $d\eta/d\xi$ of the corresponding curve at the point P_y are connected by the linear fractional relation

$$d\eta/d\xi=(\eta_u+\eta_v dv/du)/(\xi_u+\xi_v dv/du)\ , \tag{28}$$

which follows immediately from equations (27). Therefore four curves at the point P_x have tangents whose cross ratio is equal to that of the tangents of the corresponding curves at the point P_y. Since the correspondence between the pencils of tangents is not only one-to-one but also preserves cross ratio, it is a projectivity, as was to be shown.

It is possible to obtain for the point correspondence under consideration a simpler analytic representation than that appearing in equations (27). If the expressions for ξ, η as functions of u, v given in (27) are substituted in the parametric vector equation of the surface S_y, then corresponding points P_x, P_y will have the same curvilinear coordinates u, v. Therefore *it is no restriction on the correspondence or on the surfaces to suppose that the parameters have been chosen so that corresponding points have the same curvilinear coordinates.* We shall suppose from now on that this choice of parameters has been made; then, as equation (28) shows, *corresponding curves will have the same direction dv/du at corresponding points.*

A restriction will now be imposed on the correspondence itself. *Let us suppose that the tangent planes at corresponding points P_x, P_y of the two surfaces S_x, S_y intersect in straight lines. Let us further suppose that the line h of intersection of the tangent planes at each pair of points P_x, P_y passes through neither P_x nor P_y.* In this case x, y satisfy two equations which can be written in the form

$$\begin{cases} y_u=fx+mx_u+sx_v+Ay\ , \\ y_v=gx+tx_u\ +nx_v+By \qquad\qquad (mn-st\neq 0)\ . \end{cases} \tag{29}$$

For, the first of these equations asserts that the point y_u-Ay on the u-tangent at the point P_y is in the tangent plane at the point P_x of the surface S_x, and the second equation makes a similar statement involving the v-tangent at P_y. The existence of the inequality $mn-st\neq 0$ can be demonstrated by supposing that $mn-st=0$, eliminating x_u and x_v from equations (29), and thus proving that the line h passes through the point P_x, contrary

to hypothesis. In the presence of this inequality, equations (29) can be solved for x_u, x_v as linear combinations of y, y_u, y_v, x and so it becomes evident that the surfaces S_x, S_y are actually being treated in an essentially symmetrical way, although, to be sure, x and y do not enter equations (29) symmetrically.

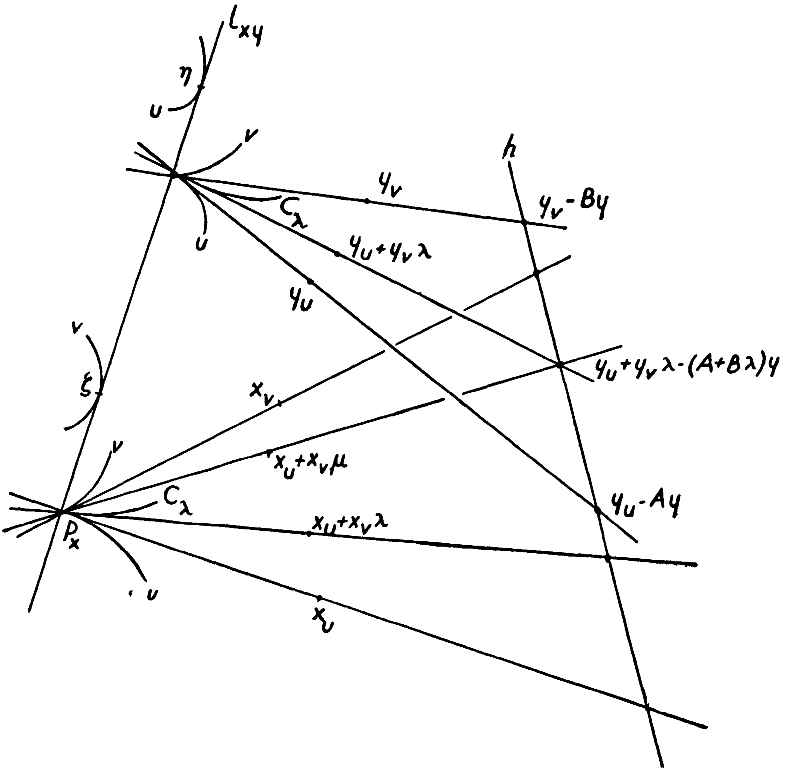

FIG. 28

The two pencils of tangents at two corresponding points P_x, P_y have been shown to be projectively related. Hence *these pencils determine on the line h a projectivity*, which we proceed to discuss. Referring to Figure 28, let us consider on the surface S_x a family of curves $dv - \lambda du = 0$, and let us denote by C_λ the curve of this family that passes through P_x. The tangent of C_λ at P_x is determined by x and $x_u + x_v\lambda$. There is a corresponding curve C_λ through the point P_y on the surface S_y. The tangent of this curve at P_y

is determined by y and $y_u+y_v\lambda$. But if the second of equations (29) is multiplied by λ and added to the first the result can be written in the form

$$y_u+y_v\lambda-(A+B\lambda)y=(f+g\lambda)x+(m+t\lambda)x_u+(s+n\lambda)x_v \ .$$

Here we have two equal expressions for the point in which the tangent of C_λ at P_y meets the line h. The direction μ of the line joining the point P_x to this point is the ratio of the coefficient of x_v to the coefficient of x_u in the right member of the equality above, and is therefore connected with the direction λ by the equation

$$(m+t\lambda)\mu=s+n\lambda \ . \tag{30}$$

This equation represents the aforesaid projectivity on the line h, since this projectivity is a section of the projectivity between the pencil of tangents of curves through the point P_x and the superposed pencil of lines joining P_x to the points where the corresponding tangents at the point P_y meet the line h.

The double points of the projectivity on the line h are found by setting $\mu=\lambda$ *and solving the equation*

$$t\lambda^2+(m-n)\lambda-s=0 \ . \tag{31}$$

We shall confine our attention from now on to the case in which $(m-n)^2+4st\neq 0$, that is, the case in which the double points are distinct.

Each double point of the projectivity on the line h is the point of intersection of a tangent at the point P_x and the corresponding tangent at the point P_y. This fact indicates that *there are on the surface S_x two one-parameter families of curves such that at every point P_x the tangent of each of the two curves of these families intersects the tangent of the corresponding curve drawn at the corresponding point P_y.* Without any additional restriction on the correspondence between the surfaces S_x, S_y these curves can be made to be the parametric curves by a suitably chosen transformation of parameters

$$\bar{u}=\varphi(u, v) \ , \qquad \bar{v}=\psi(u, v) \qquad (J=\varphi_u\psi_v-\varphi_v\psi_u\neq 0) \ .$$

The effect of this transformation on system (29) is to produce another system of the same form, of which the coefficients $\bar{s}$, $\bar{t}$ are found to be given by

$$J\bar{s}=-t\psi_u^2+(m-n)\psi_u\psi_v+s\psi_v^2 \ ,$$
$$J\bar{t}=t\varphi_u^2-(m-n)\varphi_u\varphi_v-s\varphi_v^2 \ .$$

Let us choose for φ, ψ two functionally independent solutions of the partial differential equation

$$t\theta_u^2-(m-n)\theta_u\theta_v-s\theta_v^2=0 \ .$$

Then we shall have $\bar{s}=\bar{t}=0$. This transformation amounts to setting $s=t=0$ in equations (29), so that the u-tangents and v-tangents at corresponding points P_x, P_y may intersect.

With the analytic simplification just accomplished, *the invariant* of the projectivity* on the line h, that is, the cross ratio of the invariant points and any two corresponding points, is found, by calculating the cross ratio $(\infty, 0, \lambda, \mu)$, to be n/m.

The lines l_{xy} joining pairs of corresponding points P_x, P_y certainly constitute a two-parameter family; this family can be shown to be a congruence, in the following familiar way. One starts with an arbitrary point P_z defined by

$$z=y+kx \qquad (k \text{ scalar})$$

on a line l_{xy}; one considers an arbitrary curve through the point P_x on the surface S_x; one demands that the ruled surface of lines l_{xy} intersecting S_x in this curve be developable, and that P_z be the corresponding focal point of l_{xy}; thus one readily obtains by means of equations (29) the following *equations for determining the developables and focal surfaces of the congruence xy:*

$$\begin{aligned} sdu^2-(m-n)dudv-tdv^2&=0 \ , \\ k^2+(m+n)k+mn-st &=0 \ . \end{aligned}$$

When $s=t=0$, $m\neq n$, the developables of the congruence xy are determinate and intersect the surfaces S_x, S_y in the parametric curves thereon. The foci of a line l_{xy} are then the points η, ζ given by

$$\eta=y-mx \ , \qquad \zeta=y-nx \ .$$

We may therefore state the following theorem.

If two surfaces S_x, S_y have their points in a one-to-one correspondence such that the tangent planes of the surfaces at each pair of corresponding points P_x, P_y intersect in a straight line (variable with P_x, P_y) which passes through neither P_x nor P_y and whose projectivity has two distinct double points, then the lines l_{xy} joining pairs of corresponding points form a congruence.

The converse theorem that two transversal surfaces of a congruence are such that the tangent planes of the two surfaces, at the two points where

* Graustein, 1926. 5.

any generator of the congruence pierces them, intersect in a straight line is readily demonstrated geometrically. It is sufficient to observe that each of the two developables of the congruence containing the generator intersects the surfaces in curves whose tangents at the points on the generator are coplanar.

The condition that the tangent planes at corresponding points of two surfaces S_x, S_y intersect in lines on each of which the projectivity has two distinct double points has just been shown to be equivalent to the condition that S_x, S_y be transversal surfaces of a congruence, and hence specializes, or imposes a restriction on, these surfaces. In fact, *if the parametric curves on S_x do not form a conjugate net when $s=t=0$, it can be shown that S_x is an integral surface of a system of two partial differential equations of the third order* (see Ex. 23 of Chap. IV), *and similarly for S_y.* To find these equations for the surface S_x let us take $s=t=0$ in equations (29) and differentiate the first of these equations partially with respect to v and the second with respect to u and subtract. If $A_v=B_u$, the result is a Laplace equation for x, so that the parametric curves on the surface S_x form a conjugate net. If $A_v-B_u\neq 0$, the result is

$$y=Px+Qx_u+Rx_v+Sx_{uv}\,, \tag{32}$$

where the coefficients P, Q, R, S are defined by

$$\left\{\begin{array}{ll}(B_u-A_v)P=f_v-g_u+gA-fB\,, & (B_u-A_v)Q=m_v-mB-g\,,\\ (B_u-A_v)S=m-n\,, & (B_u-A_v)R=-n_u+nA+f\,.\end{array}\right. \tag{33}$$

Differentiating equation (32) and eliminating y_u, y_v, y by means of equations (29), (32) we obtain the aforementioned equations of the third order,

$$\left\{\begin{aligned}Sx_{uuv}&=(AP+f-P_u)x+(AQ+m-P-Q_u)x_u+(AR-R_u)x_v\\&\qquad+(AS-S_u-R)x_{uv}-Qx_{uu}\,,\\ Sx_{uvv}&=(BP+g-P_v)x+(BQ-Q_v)x_u+(BR+n-P-R_v)x_v\\&\qquad+(BS-S_v-Q)x_{uv}-Rx_{vv}\,.\end{aligned}\right. \tag{34}$$

The integrability conditions of system (34), which are obtained from the equation $(x_{uuv})_v=(x_{uvv})_u$, can be shown by means of equations (33) to be satisfied identically. It is also worthy of remark that the space $S(3,0)$

at each point P_x of the surface S_x (defined to be the ambient of the osculating space S_3 at P_x of every curve on S_x through P_x) is at most a space S_7 instead of being the usual space S_9, since equations (34) are two relations among the ten points

$$x,\ x_u,\ x_v,\ x_{uu},\ x_{uv},\ x_{vv},\ x_{uuu},\ x_{uuv},\ x_{uvv},\ x_{vvv}$$

which determine the space $S(3, 0)$.

37. Pairs of surfaces in S_3. Since in ordinary space S_3 two distinct planes always intersect in a straight line, it follows that the general analytic point transformation between two analytic surfaces in space S_3 is of the type considered in the latter part of the preceding section. It is the purpose of the present section to lay the foundations for the study of *the configuration composed of two surfaces in the same space S_3 with their points in one-to-one correspondence.* The completely integrable system of partial differential equations employed for this purpose will be of a relatively general nature. The reader should presently observe for himself that several of the systems of equations that are fundamental for theories that have been developed earlier in this book are special cases of the system used here. The notation will be chosen with a view of exhibiting the relations of these systems of equations.

We now establish *the basic system of differential equations* for the configuration under consideration, and write the integrability conditions therefor. If two surfaces S_x, S_y in ordinary space have their points in a one-to-one correspondence, such that corresponding points P_x, P_y have the same curvilinear coordinates u, v, and such that each point P_y does not lie in the tangent plane of S_x at the corresponding point P_x, then S_x, S_y are a pair of integral surfaces of a system of differential equations of the form

$$(35)\qquad \left\{\begin{aligned} x_{uu} &= px+\alpha x_u+\beta x_v+Ly\ ,\\ x_{uv} &= cx+ax_u+bx_v+My\ ,\\ x_{vv} &= qx+\gamma x_u+\delta x_v+Ny\ ,\\ y_u\ &= fx+mx_u+sx_v+Ay\ ,\\ y_v\ &= gx+tx_u\ +nx_v+By\ . \end{aligned}\right.$$

In fact, if each pair of coordinates x, y is substituted in turn in the first of equations (35) with the coefficients p, α, β, L regarded as unknown, the result is four linear algebraic equations which can be solved uniquely for these coefficients. Similarly the other coefficients can be determined. *The inte-*

grability conditions for this system of equations are found by the usual method; we have

$$(x_{uu})_v=(x_{uv})_u\ , \qquad (x_{uv})_v=(x_{vv})_u\ , \qquad (y_u)_v=(y_v)_u\ ,$$

but there can exist no linear relation with non-vanishing coefficients connecting x, x_u, x_v, y, since the point y does not lie in the tangent plane at the point x. The resulting twelve integrability conditions can be written as follows:

$$(36)\qquad \left\{\begin{aligned}
a_u+ab+c+mM&=\alpha_v+\beta\gamma+tL\ ,\\
b_v+ab+c+nM&=\delta_u+\beta\gamma+sN\ ,\\
b_u+b^2+a\beta+sM&=\beta_v+b\alpha+\beta\delta+p+nL\ ,\\
a_v+a^2+b\gamma+tM&=\gamma_u+a\delta+\gamma\alpha+q+mN\ ,\\
c_u+bc+ap+fM&=p_v+c\alpha+q\beta+gL\ ,\\
c_v+ac+bq+gM&=q_u+c\delta+p\gamma+fN\ ,\\
M_u+aL+(b+A)M&=L_v+BL+\alpha M+\beta N\ ,\\
M_v+bN+(a+B)M&=N_u+AN+\delta M+\gamma L\ ,\\
t_u+t\alpha+an+mB+g&=m_v+am+s\gamma+tA\ ,\\
s_v+s\delta+bm+nA+f&=n_u+bn+t\beta+sB\ ,\\
g_u+pt+cn+fB&=f_v+qs+cm+gA\ ,\\
B_u+tL+nM&=A_v+sN+mM\ .
\end{aligned}\right.$$

It is desirable to treat the surfaces S_x, S_y in an essentially symmetrical way, although x and y do not enter equations (35) symmetrically. Let us define a function Δ by placing

$$\Delta=mn-st\ ,$$

and let us suppose that $\Delta\neq 0$. This amounts to supposing that each point P_x is not in the tangent plane of the surface S_y at the corresponding point P_y. Then x, y satisfy a system of equations of the form (35) but with the rôles of x and y interchanged. In order to calculate the coefficients of this system we observe that two of the desired equations result at once from solving the last two of equations (35) for x_u and x_v. The other three of the new equations come from differentiating the last two of (35) and eliminating the derivatives of x. The twenty coefficients of the five equations thus calculated are indicated by accents and given by the following formulas:

$$
(37)\qquad \left\{
\begin{array}{l}
\Delta f' = sB - nA\ , \quad \Delta m' = n\ , \quad \Delta s' = -s\ , \quad \Delta A' = sg - nf\ , \\
\Delta g' = tA - mB\ , \quad \Delta t' = -t\ , \quad \Delta n' = m\ , \quad \Delta B' = tf - mg\ , \\
p' = A_u + sM + mL + f'c_{11} + g'c_{12}\ , \quad \alpha' = A + m'c_{11} + t'c_{12}\ , \\
\beta' = s'c_{11} + n'c_{12}\ , \quad L' = f_u + mp + cs + A'c_{11} + B'c_{12}\ , \\
q' = B_v + tM + nN + g'c_{21} + f'c_{22}\ , \quad \gamma' = t'c_{21} + m'c_{22}\ , \\
\delta' = B + n'c_{21} + s'c_{22}\ , \quad N' = g_v + nq + ct + B'c_{21} + A'c_{22}\ , \\
c' = A_v + mM + sN + f'c_{31} + g'c_{32} = B_u + nM + tL + g'c_{41} + f'c_{42}\ , \\
a' = m'c_{31} + t'c_{32} = B + t'c_{41} + m'c_{42}\ , \\
b' = A + s'c_{31} + n'c_{32} = n'c_{41} + s'c_{42}\ , \\
M' = f_v + cm + qs + A'c_{31} + B'c_{32} = g_u + cn + pt + B'c_{41} + A'c_{42}\ ,
\end{array}
\right.
$$

where the coefficients c_{ij} are defined by placing

$$
\begin{array}{ll}
c_{11} = m_u + f + m\alpha + as\ , & c_{12} = s_u + m\beta + bs\ , \\
c_{21} = n_v + g + n\delta + bt\ , & c_{22} = t_v + n\gamma + at\ , \\
c_{31} = m_v + am + s\gamma\ , & c_{32} = s_v + f + bm + s\delta\ , \\
c_{41} = n_u + bn + t\beta\ , & c_{42} = t_u + g + an + t\alpha\ .
\end{array}
$$

Precisely as in the preceding section the parameters can be chosen so that $s = t = 0$; then equations (36), (37) are materially simplified, but it will be left to the reader to observe just what the simplifications are. The developables and focal surfaces of the congruence xy are determined by the same equations as in the preceding section.

When $s = t = 0$, the line h of intersection of the tangent planes at two corresponding points P_x, P_y of the surfaces S_x, S_y joins the points P_ρ, P_σ defined by

$$
(38)\qquad \rho = x_u + fx/m\ , \qquad \sigma = x_v + gx/n\ ,
$$

as is seen on inspecting the last two of equations (35). When u, v vary, the line h generates a congruence $\rho\sigma$, whose developables and focal surfaces will now be determined. If, as the point P_x describes a curve of the family $dv - \lambda du = 0$ on the surface S_x, the line h generates a developable of the congruence $\rho\sigma$, and if the point P_ζ defined by

$$
\zeta = \rho + k\sigma \qquad (k \text{ scalar})
$$

is the corresponding focal point of the line h, then h is tangent to the locus of the point P_ζ; consequently the derivative ζ' may be expressed as a linear

combination of ρ, σ only. But by actual calculation it is found that ζ' appears as a linear combination of x, ρ, σ, y. Setting equal to zero the coefficients of x, y therein, we obtain conditions on the functions k, λ necessary and sufficient that the line h may generate a developable of the congruence $\rho\sigma$ and have P_ζ for focal point, namely,

$$\left\{\begin{aligned} nL'+mM'k+nM'\lambda+mN'k\lambda=0\ , \\ L+Mk+M\lambda+Nk\lambda=0\ . \end{aligned}\right. \tag{39}$$

Elimination of k and substitution of dv/du for λ give *the differential equation of the developables* of the congruence* $\rho\sigma$, namely,

$$\left\{\begin{aligned} &(mLM'-nML')du^2+[m(LN'+MM')-n(NL'+MM')]dudv \\ &\qquad\qquad\qquad\qquad\qquad +(mMN'-nNM')dv^2=0\ . \end{aligned}\right. \tag{40}$$

Moreover, elimination of λ gives *the equation for the determination of the foci of the line* h, namely,

$$\left\{\begin{aligned} &m(MN'-NM')k^2+[m(LN'-MM')-n(NL'-MM')]k \\ &\qquad\qquad\qquad\qquad\qquad -n(ML'-LM')=0\ . \end{aligned}\right. \tag{41}$$

For further discussion of this subject the reader is referred to Exercises 12, 13, 14, 26.

38. Point correspondence between two surfaces in different spaces S_3. In the preceding section the two surfaces S_x, S_y considered were supposed to be in the same ordinary space S_3, and the emphasis was on the configuration determined by the two surfaces and by the correspondence between them. In the present section one of two surfaces with their points in one-to-one correspondence will be supposed to be in a space S_3, and the other also in a space S_3, ordinarily but not necessarily distinct from the first space. Moreover, the emphasis will be on the correspondence itself.

The exposition will follow in some of its main features a memoir† of Bompiani. Other geometers, notably Čech and Miss Sperry, have also contributed to the results to be set forth, which are of the same general nature as the result reached near the beginning of Section 36 concerning the projectivity between the pencils of tangents at corresponding points of the two surfaces. As that result concerns the pencils of tangent lines of corresponding curves at corresponding points, so the results here concern the bundles of osculating planes of corresponding curves at corresponding

* Grove, 1928. 2.

† Bompiani, 1923. 1.

points of the two surfaces. Near the close of this section we arrive at the particular point transformation between two surfaces that is called *projective applicability* and was first defined by Fubini.

We shall now establish *the fundamental differential equations.* Let us consider in a space S_3 a surface S_x, and in a space S_3, not necessarily the same as the first space, a surface S_y. Let the points P_x, P_y of these surfaces be in one-to-one correspondence, and let corresponding points have the same curvilinear coordinates u, v. Let us suppose that the parametric curves on the surface S_x do not form a conjugate net, so that the four coordinates x do not satisfy an equation of Laplace. Then these coordinates are solutions of a completely integrable* system of differential equations of the form

$$\left\{\begin{aligned} x_{uu} &= px + \alpha x_u + \beta x_v + L x_{uv} \ , \\ x_{vv} &= qx + \gamma x_u + \delta x_v + N x_{uv} \ . \end{aligned}\right. \tag{42}$$

For, if the four coordinates x are substituted one at a time in the first of equations (42) with the coefficients p, α, β, L regarded as unknown, the result is four linear algebraic equations which can be solved for these coefficients. The coefficients q, γ, δ, N can be determined in like manner. Similarly, if the parametric curves on the surface S_y do not form a conjugate net, the four coordinates y are solutions of a system of the same form, whose coefficients will be indicated by accents.

The correspondence between the points of the surfaces S_x, S_y determines a correspondence between the bundles of planes with centers at two corresponding points P_x, P_y. The equations of this correspondence can be found in the following way. Let us consider the curve C_λ of the family $dv - \lambda du = 0$ through the point P_x on the surface S_x. The coordinates ξ of the osculating plane of C_λ at P_x, referred to the tetrahedron x, x_u, x_v, x_{uv} with suitably chosen unit point, are found to be given by

$$\left\{\begin{aligned} &\xi_1 = 0 \ , \qquad \xi_2 = \lambda(L + 2\lambda + N\lambda^2) \ , \qquad \xi_3 = -(L + 2\lambda + N\lambda^2) \ , \\ &\xi_4 = \lambda' + \beta - \alpha\lambda + \delta\lambda^2 + \gamma\lambda^3 \qquad\qquad (\lambda' = \lambda_u + \lambda\lambda_v) \ . \end{aligned}\right. \tag{43}$$

The coordinates η of the osculating plane of the corresponding curve at P_y on S_y, referred to the tetrahedron y, y_u, y_v, y_{uv} with suitably chosen unit point, are given by similar formulas with the same λ, λ' but with accents on the coefficients. Homogeneous elimination of λ, λ' furnishes *the equations of the correspondence between the bundles of planes with centers at corresponding*

* Green, 1920. 2.

points P_x, P_y, corresponding planes being the osculating planes of corresponding curves:

$$(44)\begin{cases}\eta_1=0\ , \quad \eta_2=\xi_2(L'\xi_3^2-2\xi_3\xi_2+N'\xi_2^2)\ , \quad \eta_3=\xi_3(L'\xi_3^2-2\xi_3\xi_2+N'\xi_2^2)\ ,\\ \eta_4=\xi_4(L\xi_3^2-2\xi_3\xi_2+N\xi_2^2)+(\beta-\beta')\xi_3^3+(\alpha-\alpha')\xi_3^2\xi_2\\ \qquad\qquad\qquad\qquad\qquad +(\delta-\delta')\xi_3\xi_2^2+(\gamma-\gamma')\xi_2^3\ .\end{cases}$$

The equations of the inverse transformation can be obtained by keeping in mind that $\eta_1=0$ and solving equations (44) for the ratios of η_2, η_3, η_4, or more simply by merely interchanging in (44) the plane coordinates ξ and η and at the same time the accented and unaccented letters. Therefore *the correspondence between the bundles of planes at corresponding points of the two surfaces is a cubic Cremona transformation.*

Considerable analytic simplification can be obtained by supposing that the parametric curves on the surface S_y are the asymptotic curves, so that $L'=N'=0$. Under this assumption *let us seek for those planes at a point P_x whose corresponding planes at the corresponding point P_y are indeterminate.* For this purpose we set $\eta_2=\eta_3=\eta_4=0$ in equations (44) and solve the resulting equations for the ratios of ξ_2, ξ_3, ξ_4. As the result we find three planes whose coordinates are

$$(0,0,0,1)\ , \qquad (0,N,0,\gamma'-\gamma)\ , \qquad (0,0,L,\beta'-\beta)\ .$$

If the asymptotic curves on the surfaces S_x, S_y do not correspond, then $LN\neq 0$ and these three planes are distinct. The first is the tangent plane at the point P_x of the surface S_x. By means of the equations of the inverse of the transformation (44) it can be shown that this plane corresponds to every plane through either of the straight lines

$$y_4=Ly_2^2+2y_2y_3+Ny_3^2=0\ ,$$

which are the tangents at the point P_y of the curves on the surface S_y corresponding to the asymptotic curves on the surface S_x. Similarly, it can be shown that the plane $(0, N, 0, \gamma'-\gamma)$ corresponds to every plane through the v-tangent, $y_4=y_2=0$, at the point P_y, and that the plane $(0, 0, L, \beta'-\beta)$ corresponds to every plane through the u-tangent, $y_4=y_3=0$, at P_y. It will be left to the reader to show that the last two planes at the point P_x correspond to the osculating planes of curves on S_y having inflexions at P_y. Finally, the line of intersection of these two planes at P_x passes, of course, through P_x and passes also through the point

$$[0,\ L(\gamma-\gamma'),\ N(\beta-\beta'),\ LN]\ .$$

This line is called *the axis of the correspondence at the point P_x.*

We next define a projectivity between the bundles of lines at corresponding points P_x, P_y of the two surfaces S_x, S_y. Any line l_{yz} through the point P_y may be regarded (see Fig. 29) as joining P_y to a point P_z with coordinates

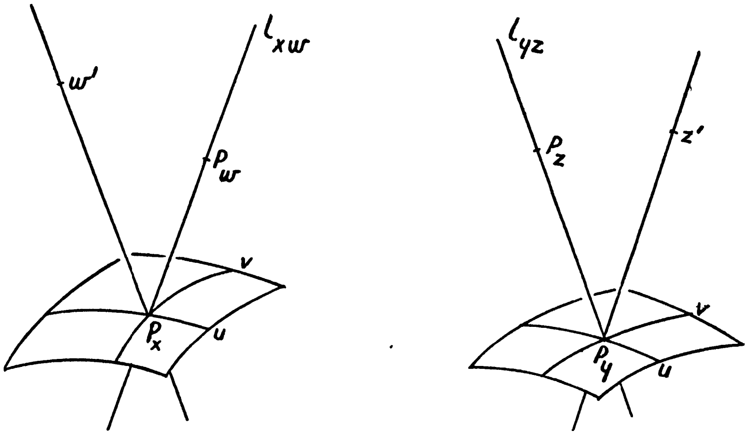

FIG. 29

$0, z_2, z_3, z_4$. To the planes of the pencil having l_{yz} as axis correspond at the point P_x the planes enveloping the cone whose equations, written by the aid of the condition of united position $z_2\eta_2+z_3\eta_3+z_4\eta_4=0$ and equations (44), are

$$\text{(45)}\quad \begin{cases} \xi_1 = z_4\xi_4(L\xi_3^2-2\xi_3\xi_2+N\xi_2^2)+z_4(\beta-\beta')\xi_3^3+z_4(\gamma-\gamma')\xi_2^3 \\ \qquad +[z_4(\alpha-\alpha')-2z_3]\xi_3^2\xi_2+[z_4(\delta-\delta')-2z_2]\xi_3\xi_2^2=0\ . \end{cases}$$

This cone has three cusp-planes which intersect in the line l_{xw} joining P_x to the point P_w whose coordinates $0, w_2, w_3, w_4$ are given by

$$\text{(46)}\quad \begin{cases} w_2=4z_2+2Nz_3+[2(\delta'-\delta)+N(\alpha'-\alpha)+3L(\gamma'-\gamma)]z_4\ , \\ w_3=2Lz_2+4z_3+[2(\alpha'-\alpha)+L(\delta'-\delta)+3N(\beta'-\beta)]z_4\ , \\ w_4=4(1-LN)z_4\ . \end{cases}$$

These equations evidently represent a projectivity between points P_z in the plane $z_1=0$ and points P_w in the plane $w_1=0$. Projecting these planes from the points P_x and P_y respectively, we obtain two bundles of lines with centers at P_x, P_y; the equations (46) may be said to represent *a projectivity between the lines of these bundles.*

The projectivity (46) is degenerate if, and only if,

$$(1-LN)(4-LN)=0 .$$

Let us with Bompiani consider these two possibilities briefly and then exclude them from our future considerations. If LN=1, the surface S_x is developable. If $LN=4$, the cross ratio of the two parametric tangents at the point P_x (which correspond to the asymptotic tangents at the point P_y) and the two asymptotic tangents at P_x, in one of the possible orders, is $(-1+i3^{1/2})/2$, where $i^2=-1$. Therefore *if* $LN=4$ *the parametric tangents at the point* P_x, *one asymptotic at* P_x, *and the harmonic conjugate of the other asymptotic tangent with respect to the parametric tangents form an equianharmonic group.*

We now define with Bompiani a certain *homology in the bundle of lines at a point* P_x. Let us consider the projectivity corresponding to the projectivity (46) when the rôles of x and y are interchanged, in which a line xw' through the point P_x, regarded as the axis of a pencil of planes, corresponds to a line yz' through the corresponding point P_y, regarded as the cusp-axis of a cone. The equations of this projectivity are

$$(47) \qquad \begin{cases} z_2'=-2w_2'+Nw_3'+(\delta'-\delta)w_4' , \\ z_3'=Lw_2'-2w_3'+(\alpha'-\alpha)w_4' . \\ z_4'=-2w_4' . \end{cases}$$

Let us suppose that the line yz' coincides with the line yz. Then z_2', z_3', z_4' are proportional to z_2, z_3, z_4 respectively, and these coordinates can be eliminated from equations (46), (47). The result of the elimination is

$$(48) \qquad \begin{cases} w_2=(4-LN)w_2'+3L(\gamma'-\gamma)w_4' , \\ w_3=(4-LN)w_3'+3N(\beta'-\beta)w_4' , \\ w_4=4(1-LN)w_4' . \end{cases}$$

These equations evidently represent a projectivity in the bundle of lines at the point P_x, and this projectivity can be shown to be a homology by showing that it has a flat pencil of invariant lines and one other invariant line through the center of the pencil. Thus the reader may demonstrate the following theorem. *All the tangent lines at the point* P_x *are invariant lines of this homology; the other invariant line is the axis of the correspondence at* P_x.

The theory of union curves* can be advantageously employed here. The

* Sperry, 1918. 2, p. 214.

differential equation of the union curves on the surface S_x of the congruence xw generated by the line joining the point P_x to the point $(0, w_2, w_3, 1)$ can be written by the aid of the condition of united position $w_2\xi_2+w_3\xi_3+\xi_4=0$ and equations (43). Thus we obtain

$$\lambda' = -\beta+Lw_3+(\alpha-Lw_2+2w_3)\lambda-(\delta+2w_2-Nw_3)\lambda^2+(\gamma-Nw_2)\lambda^3 , \tag{49}$$

wherein it is understood that λ now stands for dv/du and λ' for d^2v/du^2. Similarly, the equation of the union curves on the surface S_y of the congruence yz generated by the line joining the point P_y to the point $(0, z_2, z_3, 1)$ is

$$\mu' = -\beta'+(\alpha'+2z_3)\mu-(\delta'+2z_2)\mu^2+\gamma'\mu^3 , \tag{50}$$

wherein $\mu=dv/du$ and $\mu'=d^2v/du^2$. If to every curve (49) corresponds a curve (50), and vice versa, these two equations must give $\lambda'=\mu'$ whenever $\lambda=\mu$. Conditions necessary and sufficient therefor are

$$\begin{cases} \beta-\beta'=Lw_3 , \qquad \gamma-\gamma'=Nw_2 , \\ \alpha-\alpha'-2z_3=Lw_2-2w_3 , \\ \delta-\delta'-2z_2=-2w_2+Nw_3 . \end{cases} \tag{51}$$

If, further, none of the asymptotic curves correspond on the two surfaces, then $LN\neq0$ and the congruence xw consists of the axes of the correspondence at the points of the surface S_x, while the congruence yz consists of the axes of the correspondence at the points of the surface S_y. But if the union curves on the surface S_x of *every* congruence xw correspond to the union curves on the surface S_y of *some* congruence yz, then the first two of equations (51) are identities in w_3, w_2, and we have the conditions

$$L=N=\beta-\beta'=\gamma-\gamma'=0 . \tag{52}$$

Under these conditions not only do the asymptotic curves correspond on the surfaces S_x and S_y but these two surfaces are related so that they are, as we shall see in the next paragraph, *projectively* applicable.*

Projective applicability of two surfaces was first defined and studied by Fubini. Later Čech proved† that Fubini's definition was equivalent to saying that *two surfaces are projectively applicable in case their points can be placed in a one-to-one correspondence such that the bundle of planes at each point of one surface is projective with the bundle of planes at the corresponding*

* Fubini and Čech, 1926. 1, pp. 118–24.

† Čech, 1922. 8.

point of the other surface, when corresponding planes are the osculating planes of corresponding curves at these points. Fubini's definition is also equivalent to the following characterization. *Two surfaces are projectively applicable in case there is a one-to-one correspondence between their points such that* to all curves through a point on one surface with their osculating planes at this point forming a pencil correspond curves through the corresponding point on the other surface with their osculating planes at this point also forming a pencil.* We arrived at this property at the close of the preceding paragraph. It follows that at corresponding points of corresponding curves on two projectively applicable surfaces the differential invariant

$$(\beta du^3+\gamma dv^3)/du dv$$

(see § 24) is the same, just as in the theory of metrically applicable surfaces the differential invariant $Edu^2+2Fdudv+Gdv^2$ representing the squared element of arc length is the same at corresponding points of corresponding curves (see § 44, Chap. VI). Just as the first fundamental coefficients E, F, G at corresponding points of two metrically applicable surfaces are the same, so the coefficients β, γ at corresponding points of two projectively applicable surfaces are the same, and the projective differential invariant above is appropriately called *the projective linear element.*

39. Quadratic nets and congruences, and the transformation of Ribaucour. We begin by stating some definitions. A conjugate net in a space S_n is called *a quadratic net* in case the net lies entirely on a non-singular hyperquadric; similarly, a congruence of lines on a non-singular hyperquadric is called *a quadratic congruence.* Such a congruence should not be confused with an algebraic congruence of the second degree. Two quadratic nets on the same hyperquadric and in the relation of a fundamental transformation are said to correspond by a *transformation of Ribaucour.* This transformation will be studied briefly in this section.

The analytic criteria for a quadratic net now engage our attention. Let us consider a parametric quadratic net N_x in a space S_n. The coordinates x of a variable point P_x on the net N_x not only satisfy a Laplace equation of the form (1) but also satisfy a homogeneous quadratic equation in $n+1$ variables. Since a non-singular quadratic form in $n+1$ variables can be reduced by a projective transformation to the sum of $n+1$ squares, the latter equation can be written with customary abbreviation in the form

$$\Sigma x^2=0\,. \tag{53}$$

Consequently, *equations* (1), (53) *represent analytically a quadratic net.*

* Fubini and Čech, 1926. 1, p. 122.

We shall now prove that *if a congruence is conjugate to a quadratic net and if each line of the congruence meets the hyperquadric on which the net lies in just two distinct points, then the second intersection of the congruence and the hyperquadric is also a net conjugate to the congruence.* These two nets are F transforms of each other, and such a transformation F is by definition a *transformation* of Ribaucour.* For the purpose of the proof let us consider a quadratic net N_x with x satisfying equations (1), (53). Let us also consider a congruence $\eta\zeta$ with focal nets N_η, N_ζ and conjugate to the net N_x; let η satisfy the equations

$$\eta_u=(\varphi_u+b\varphi)x\ ,\qquad \eta_v=\varphi(x_v-ax)\ , \tag{54}$$

where φ is a solution of the adjoint of equation (1), according to the conclusion of Section 32. The point y defined by

$$y=\eta+\lambda x \qquad\qquad (\lambda \text{ scalar})$$

is on the line $x\eta$, and this point is also on the hyperquadric (53) in case $\Sigma y^2=0$. In view of the fact that x satisfies equation (53), we find that the line $x\eta$ meets the hyperquadric (53) in the point x, of course, and in the point y uniquely determined by solving for λ the equation

$$Y+\lambda X=0\ , \tag{55}$$

in which *the functions X, Y are defined by placing*

$$X=2\Sigma x\eta\ ,\qquad Y=\Sigma\eta^2 \qquad\qquad (XY\neq 0)\ .$$

The intersection point y is therefore given by the formula

$$y=X\eta-Yx\ , \tag{56}$$

the proportionality factor for y having been suitably chosen. It is not difficult to show that *X, Y satisfy equations* (54) *when substituted in place of x, η respectively.* It follows that *X is a solution of equation* (1), while *Y is a solution of the Laplace equation satisfied by η*, namely,

$$\eta_{uv}=\varphi H\eta_u/(\varphi_u+b\varphi)+(\varphi_u+b\varphi)\eta_v/\varphi\ .$$

Consequently it is easy to verify the equation

$$(\varphi_u+b\varphi)y=Y_u\eta-Y\eta_u\ .$$

* Ribaucour, 1872. 1, p. 1491.

Comparison of this equation with equation (IV, 81) shows that the point y generates a net which is a Levy transform of the net N_η and which is therefore conjugate to the congruence $x\eta$, that is, the congruence $\eta\zeta$. This completes the proof.

By definition *a congruence in space S_n is quadratic in case all of its lines are on a non-singular hyperquadric.* If a congruence is quadratic it follows at once that both of its focal nets are quadratic. But if a net is quadratic it does not necessarily follow that the congruence of tangents of the curves of one family of the net is quadratic. In fact, *the congruence of v-tangents of a quadratic net* (1), (53) *is itself quadratic if, and only if,* $\Sigma x_v^2=0$.

It can be shown that *if one focal net of a quadratic congruence is subjected to a transformation of Ribaucour, this transformation carries the congruence into a quadratic congruence.* It is sufficient to consider the quadratic net (1), (53) with $\Sigma x_v^2=0$ and to prove by direct calculation that y_v derived from equation (56) is such that $\Sigma y_v^2=0$. The details of the demonstration are omitted.

It can further be shown that *the two quadratic congruences xx_v and yy_v are harmonic to the net N_η.* It is sufficient to observe that equations (54) show that the line xx_v is the same as the line $\eta_u\eta_v$, and to prove by a little calculation that the line yy_v is the same as the line joining the points $(\eta/Y)_u$ and $(\eta/Y)_v$. Incidentally, it follows that corresponding lines xx_v and yy_v intersect in a point z which generates a net N_z that is conjugate to both congruences (see Ex. 17 of Chap. IV). Obviously this net N_z is itself quadratic.

40. The correspondence between lines of R_3 and points on a hyperquadric in S_5. We begin with a few comments on two symbols R_3 and $(a\omega)$. *Ordinary projective space with the straight line as generating element will be called ordinary ruled space and will be denoted by the symbol R_3,* in spite of the fact that this space is four-dimensional. Recalling the definition of the plückerian coordinates ω of a line, as stated in Section 7, *we define the symbol $(a\omega)$ by the formula*

$$(a\omega)=a_{34}\omega_{12}+a_{42}\omega_{13}+a_{23}\omega_{14}+a_{14}\omega_{23}+a_{13}\omega_{42}+a_{12}\omega_{34}\ . \tag{57}$$

Now the equation of a linear complex takes the simple form $(a\omega)=0$, and the quadratic relation (I, 43) satisfied by the coordinates ω can be written in the form $(\omega\omega)=0$. The linear complex is special, according to Section 20, in case $(aa)=0$.

The correspondence with which this section is concerned is defined as follows. *Let the six homogeneous coordinates ω of a line in ordinary ruled space R_3 be interpreted as the projective homogeneous coordinates of a point in a linear space S_5; then to the lines of space R_3 correspond the points on the hyperquadric*

Q_4 whose equation is $(\omega\omega)=0$ in the space S_5, corresponding point and line having the same or proportional coordinates. This correspondence, which seems to have been pointed out* first by Klein, and to have been studied extensively† first by Segre, has proved to be a fruitful transformation of problems concerning surfaces and congruences in ordinary space. Some of the essential features of this correspondence will be summarized in this section so that they will be available for use in connection with the applications of the transformation in Sections 41 and 43.

Two lines l_ω, l_τ with coordinates ω, τ in space R_3 can be shown to intersect in case $(\omega\tau)=0$, and then any linear combination $p\omega+q\tau$ gives‡ the coordinates of a line in the flat pencil determined by l_ω, l_τ. In space S_5 the corresponding points P_ω, P_τ determine a straight line which lies entirely on the hyperquadric Q_4. Hence we have the theorem:

To a flat pencil of lines in ordinary ruled space R_3 corresponds a rectilinear generator of the hyperquadric Q_4, each line of the pencil corresponding to a point of the generator.

Three lines l_ω, l_τ, l_σ in space R_3, not belonging to a flat pencil, intersect in pairs in case $(\omega\tau)=(\tau\sigma)=(\sigma\omega)=0$. In this case (see Fig. 30) the lines either belong to a ruled plane or to a bundle of lines. Then any linear combination $p\omega+q\tau+r\sigma$ gives the coordinates of a line intersecting all three lines l_ω, l_τ, l_σ and hence belonging to the ruled plane or to the bundle as the case may be. In space S_5 the corresponding points P_ω, P_τ, P_σ determine a plane which lies entirely on the hyperquadric Q_4. It is known that there are two systems of planar generators on a hyperquadric in space S_5, just as there are two systems of rectilinear generators on a quadric in space S_3. We formulate our result as follows:

To the ruled planes of space R_3 correspond the planar generators of one family of the hyperquadric Q_4, and to the bundles of lines of R_3 correspond the planar generators of the other family of Q_4.

To *a linear complex* with the equation $(a\omega)=0$ in space R_3 corresponds the intersection of the hyperquadric Q_4 by the hyperplane represented in space S_5 by the same equation. The point a, which is the pole of this hyperplane with respect to Q_4, is called *the second image* of the complex. The complex is special in case the point a is on the hyperquadric Q_4. So we have the theorem:

To the linear complexes of space R_3 correspond as second images the points of space S_5, special linear complexes in R_3 corresponding to points on the hyperquadric Q_4.

* Klein, 1872. 2, p. 261. † Segre, 1885. 1.

‡ Bertini, 1923. 2, p. 158; Graustein, 1930. 1, pp. 463–65.

The lines common to two complexes with the equations $(a\omega)=0$, $(b\omega)=0$ in space R_3 form a linear congruence which corresponds to the intersection of the hyperquadric Q_4 by the space S_3 represented in space S_5 by the same two equations, and which has for second image the polar line ab of the space S_3 with respect to Q_4. There are three possibilities as to the intersection of the line ab and the hyperquadric Q_4; accordingly, *there are three classes of linear congruences in space* R_3. First of all, the line ab may intersect the

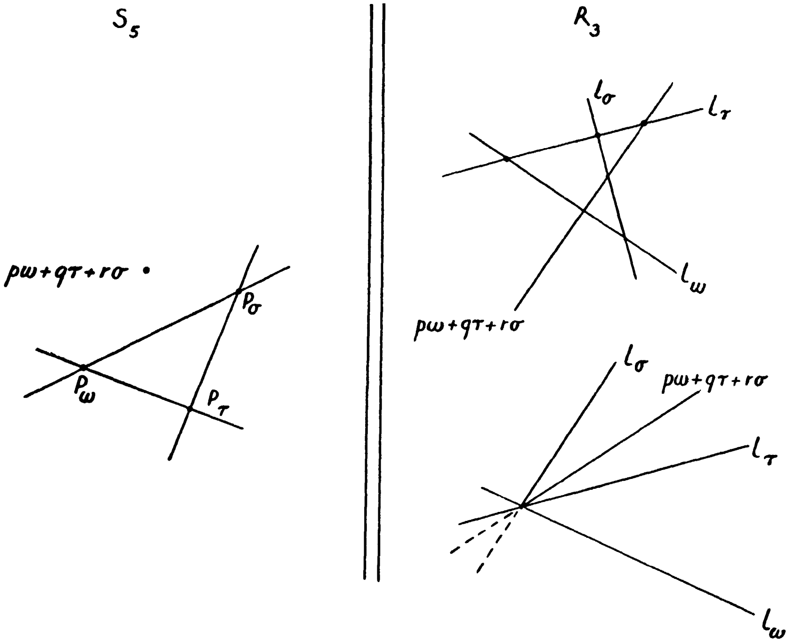

FIG. 30

hyperquadric Q_4 in just two distinct points P_1, P_2. Every point on the line ab is the second image of one of a pencil of linear complexes containing the congruence. The two points P_1, P_2 are the second images of two special linear complexes containing the congruence. Therefore the congruence has two distinct skew straight line directrices corresponding to the intersection points P_1, P_2, and consists of the straight lines intersecting these two skew lines. Secondly, in case the line ab is tangent to the hyperquadric Q_4, the congruence has two coincident directrices. Finally, if the line ab lies entirely on Q_4, all the complexes of the pencil containing the congruence are special,

their axes forming a flat pencil; the congruence consists of lines intersecting all the lines of this pencil.

Conjugate lines with respect to a linear complex may be defined as follows. Let us consider in space R_3 a non-special linear complex $(a\omega)=0$ and a line l_b. This line may be regarded as the axis of a special linear complex $(b\omega)=0$. Then the two complexes determine a pencil, and the other special complex in this pencil has for second image in space S_5 the second intersection P_c of the line ab with the hyperquadric Q_4. In space R_3 the lines l_b and l_c are said to be *conjugate*, or *polar*, with respect to the complex $(a\omega)=0$. It may be shown that the lines of a complex which intersect one of two lines conjugate with respect to the complex intersect the other also, forming a linear congruence; and all lines intersecting both conjugate lines belong to the complex.

The lines common to three complexes with the equations $(a\omega)=0$, $(b\omega)=0$, $(c\omega)=0$ in space R_3 form a configuration which will be shown to be *a regulus*, that is, to consist of lines intersecting three skew lines. The configuration corresponds to the intersection of the hyperquadric Q_4 by the plane represented in space S_5 by the same three equations, and has for second image the polar plane abc of this plane with respect to Q_4. Every point in the plane abc is the second image of one of a net, or two-parameter family, of complexes containing the configuration under discussion. Let us suppose that the points P_a, P_b, P_c are on the conic of intersection of the plane abc and Q_4, none of the lines ab, bc, ca being generators of Q_4. Then every line l_ω of the configuration in space R_3 meets the three lines l_a, l_b, l_c, and the configuration is therefore a regulus, as was to be shown. The lines linearly dependent on l_a, l_b, l_c form the other regulus on the quadric sustaining the first. The special relative positions of the plane abc and the hyperquadric Q_4 are in accord with the various types of degenerate reguli. From this discussion we have the following theorem:

Two reguli on the same quadric in space R_3 correspond to the conics of intersection of the hyperquadric Q_4 by two planes that are reciprocal polars with respect to Q_4.

To *a ruled surface* in space R_3 corresponds a curve on the hyperquadric Q_4 in space S_5. If the ruled surface is developable, then any two consecutive generators intersect determining a pencil of lines, and the tangents of the curve on the hyperquadric Q_4 are generators of Q_4.

In conclusion we shall indicate synthetically an interesting application of the transformation studied in this section. To *a non-ruled surface* S in space R_3, regarded as the envelope of the complex* of its tangents, corre-

* Bompiani, 1912. 1, p. 406.

sponds a variety V_3 on the hyperquadric Q_4. But since the complex of tangents of the surface S is composed of ∞^2 pencils of lines, the variety V_3 is composed of ∞^2 lines; this variety is, in fact, a quadratic congruence. To the congruence of lines tangent to the surface S at the points of a curve on S, which is composed of ∞^1 flat pencils of lines, corresponds a ruled surface in the congruence V_3. In particular, to the congruence of tangents of the surface S along an asymptotic curve on S, in which two consecutive pencils have a line in common, namely an asymptotic tangent, corresponds a developable in the congruence V_3, on which two consecutive generators intersect in a focal point of a generator. To the developable of tangents of an asymptotic curve on the surface S corresponds the edge of regression of a developable in the congruence V_3. To the congruence of tangents of one family of asymptotic curves on the surface S corresponds one focal surface of the congruence V_3; on this surface the edges of regression of one family of developables of V_3 represent the developables in the congruence of asymptotic tangents of S, while the curves conjugate to these edges of regression, i.e., the curves of contact of the developables of the other family of the congruence V_3, represent the ruled surfaces in the congruence of asymptotic tangents of the surface S constructed at the points of the asymptotic curves of the other family.

41. Surfaces in ordinary ruled space. The purpose of this section is to apply the transformation discussed in the preceding section to the theory of non-ruled surfaces in ordinary ruled space. The methods used are analytic rather than synthetic as they were in the preliminary discussion of the problem at the close of the last section. The surface under consideration is referred to its asymptotic curves as in Chapter III, and by means of the transformation some of the theory of Chapter IV and the earlier sections of the present chapter, particularly the theory of quadratic nets and congruences and of reciprocally polar sequences of Laplace in space S_n, can be used by placing $n=5$ to deduce results concerning the geometry of surfaces in ordinary space. These results principally concern certain complexes considered by Wilczynski, the directrices of Wilczynski, the canonical pencils, the tetrahedron of Demoulin, the curves of Darboux, and those surfaces whose asymptotic curves belong to linear complexes.

We proceed to calculate *two Laplace equations* which play a large part in this section. Let us consider in ordinary space an integral surface S of equations (III, 6). The plückerian line coordinates ω of the u-tangent, and the coordinates τ of the v-tangent, at a point P_x on the surface S are proportional to certain determinants of the second order, which can be written

with customary abbreviation in the respective forms (x, x_u) and (x, x_v). Let us choose the two proportionality factors for these coordinates so that

(58) $$\omega = e^{-\theta}(x, x_u) , \qquad \tau = e^{-\theta}(x, x_v) .$$

Differentiation and reduction by means of equations (III, 6) give

(59) $$\omega_u = \beta\tau , \qquad \tau_v = \gamma\omega ,$$

and then

(60) $$\omega_{uv} = \beta\gamma\omega + (\log\beta)_v\omega_u , \qquad \tau_{uv} = \beta\gamma\tau + (\log\gamma)_u\tau_v .$$

These are the Laplace equations which we set out to calculate. Employing the transformation of the preceding section, we may formulate some immediate consequences as follows.

The two asymptotic tangents l_ω, l_τ at a point P_x on a surface S in space R_3 correspond to two points P_ω, P_τ on the hyperquadric Q_4 in space S_5. As P_x varies on S, the points P_ω, P_τ generate two quadratic nets N_ω, N_τ on Q_4. The net N_ω is the first Laplace transform of the net N_τ. The nets N_ω, N_τ are the focal nets of the congruence V_3 which represents on Q_4 the complex of tangents of the surface S.

The coordinates ω satisfy not only the first of equations (60) but also *a linear homogeneous partial differential equation of the third order* which can be found in the following way. Making use of the first of equations (58) and the equations obtained therefrom by differentiation, and employing equations (6), . . . , (10) of Chapter III, we calculate the following formulas:

(61) $$\begin{cases} \omega_{uu} + (\log\gamma)_u\omega_u + \beta(\omega_v + \theta_v\omega) = 2\beta e^{-\theta}(x, x_{uv}) , \\ \omega_{uu} + (\log\gamma)_u\omega_u - \beta(\omega_v + \theta_v\omega) = 2\beta e^{-\theta}(x_u, x_v) , \\ \omega_{uuu} + (\log\gamma/\beta)_u\omega_{uu} + [\gamma_{uu}/\gamma - \theta_u(\log\gamma)_u - p - \pi]\omega_u \\ \qquad\qquad - \beta(\beta\gamma + \theta_{uv})\omega = 2\beta e^{-\theta}(x_u, x_{uv}) , \\ \omega_{vv} + \theta_v\omega_v + (\theta_{vv} - q - \chi)\omega - (\beta\gamma + \theta_{uv})\omega_u/\beta = 2e^{-\theta}(x_v, x_{uv}) . \end{cases}$$

Then differentiating the last of these equations with respect to v and reducing the result, we obtain the equation sought for, namely, the second equation of the following system, in which the first equation is merely the first of (60) rewritten for convenience of reference, and the functions l, m were defined in Section 16:

(62) $$\begin{cases} \omega_{uv} = \beta\gamma\omega + (\log\beta)_v\omega_u , \\ \omega_{vvv} = (\gamma/\beta)\{\omega_{uuu} - (\log\beta^2\gamma)_u\omega_{uu} + [\gamma_{uu}/\gamma + \theta_u(\log\gamma)_u \\ \qquad\qquad + \varphi_u - 2(p+\pi)]\omega_u\} + m_v\omega + 2m\omega_v . \end{cases}$$

Performing the requisite calculations would show that *the system* (62) *is completely integrable.* This system is fundamental for the study of the net N_ω that corresponds on the hyperquadric Q_4 to the congruence of asymptotic u-tangents of the surface S in ordinary ruled space R_3. A symmetric system of equations could be written for the net N_τ which corresponds to the congruence of asymptotic v-tangents of the surface S.

Some further comments on the net N_ω may be based on the first of equations (62). *The invariants H, K of the net N_ω are given by the formulas*

$$H=\beta\gamma-(\log\beta)_{uv}\ ,\qquad K=\beta\gamma\ . \tag{63}$$

The first Laplace transformed net N_1 of the net N_ω is generated by the point ω_1 defined by

$$\omega_1=\omega_v-(\log\beta)_v\omega\ .$$

By means of equations (IV, 19) the coefficients and invariants of the Laplace equation for the net N_1 are found to have the values given by

$$\begin{cases} a_1=(\log\beta H)_v\ , & b_1=0\ , \qquad c_1=H\ ,\\ H_1=H-(\log\beta H)_{uv}\ , & \qquad\qquad K_1=H\ . \end{cases} \tag{64}$$

In general, for the rth transformed net N_r of the net N_ω there is a Laplace equation whose coefficients and invariants are shown by equations (IV, 23) to have the values given by

$$\begin{cases} a_r\ =(\log\beta HH_1\ .\ .\ .\ H_{r-1})_v\ , & b_r=0\ , \qquad c_r=H_{r-1}\ ,\\ H_r=H_{r-1}-(\log\beta HH_1\ .\ .\ .\ H_{r-1})_{uv}\ , & \qquad\qquad K_r=H_{r-1}\ . \end{cases} \tag{65}$$

Similar formulas could easily be written for the net N_τ.

Let us consider a point P_ω of the net N_ω in space S_5, as shown in Figure 31. Associated with P_ω we have *six hyperplanes* determined by the following six sets of five points each:

$$\begin{array}{r} \omega,\ \omega_1,\ \omega_2,\ \omega_3,\ \omega_4\ ;\\ \tau,\ \omega,\ \omega_1,\ \omega_2,\ \omega_3\ ;\\ \tau_{-1}\ ,\tau\ ,\omega,\ \omega_1,\ \omega_2\ ;\\ \tau_{-2}\ ,\tau_{-1}\ ,\tau\ ,\omega\ ,\omega_1\ ;\\ \tau_{-3}\ ,\tau_{-2}\ ,\tau_{-1}\ ,\tau\ ,\omega\ ;\\ \tau_{-4}\ ,\tau_{-3}\ ,\tau_{-2}\ ,\tau_{-1}\ ,\tau\ . \end{array}$$

Each of these hyperplanes may be regarded as the osculating hyperplane of a v-curve on the surface generated by the first point mentioned in deter-

mining it, and also as the osculating hyperplane of a u-curve on the surface generated by the last point mentioned in determining it. Moreover, the third hyperplane is the space $S(2, 0)$, defined as in Section 27, at the point P_ω of the surface S_ω sustaining the net N_ω, while the fourth hyperplane bears the same relation to the net N_τ. It can be shown that *the poles of these six hyperplanes with respect to the hyperquadric Q_4 are respectively the six points τ_{-2}, τ_{-1}, τ, ω, ω_1, ω_2*. For this purpose it is sufficient, according to Section 33, to make the demonstration in one instance; let us choose the

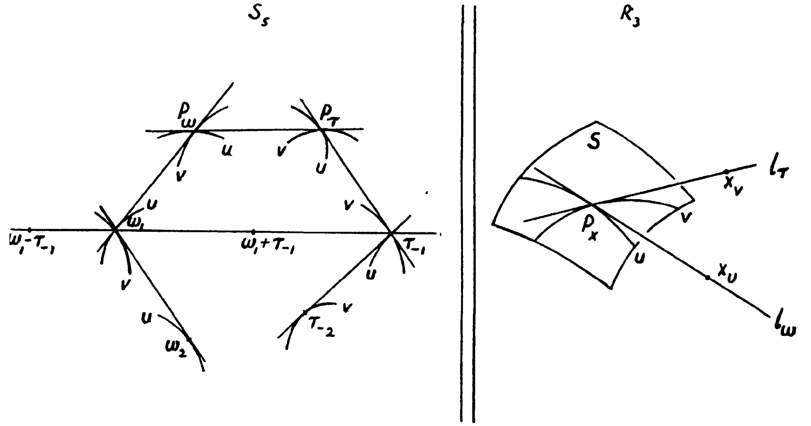

FIG. 31

third for simplicity. The polar hyperplane of the point P_τ has the equation $(\tau\xi) = 0$ in variable coordinates ξ, and it is not difficult to show that this equation is satisfied when each of τ_{-1}, τ, ω, ω_1, ω_2 is substituted in place of ξ, thus completing the proof.

The six hyperplanes just considered intersect the hyperquadric Q_4 in *six varieties V_3* which correspond to *six linear complexes in space R_3*. The intersection of Q_4 by the first hyperplane corresponds to *the osculating linear complex* L_u along the generator xx_u of the ruled surface of u-tangents constructed at the points of the v-curve through the point P_x on the surface S*, which therefore has for second image the point τ_{-2}. Similarly, *the osculating linear complex L_v* along the generator xx_v of the ruled surface of v-tangents constructed at the points of the u-curve through the point P_x on the surface S has for second image the point ω_2. *The osculating linear complexes M_v and*

* Wilczynski, 1908. 2, p. 83.

M_u at the point P_x of the v-curve and the u-curve on the surface S have for second images the points τ_{-1} and ω_1 respectively, while the points τ, ω may be regarded as the second images of *the special linear complexes whose axes are respectively the v-tangent and the u-tangent at the point P_x of the surface S.*

We shall now find *the points on the hyperquadric Q_4 which represent the directrices of Wilczynski* through a point P_x on the surface S. These are the points where the line $\tau_{-1}\,\omega_1$, which is the second image of the congruence of intersection of the linear complexes M_v, M_u, intersects the hyperquadric Q_4. Let us place

$$
(66)\qquad \begin{cases} A_3=e^{-\theta}(x,\,x_{uv})\ , & A_4=e^{-\theta}(x_u,\,x_v)\ , \\ A_5=e^{-\theta}(x_{uv},\,x_u)\ , & A_6=e^{-\theta}(x_v,\,x_{uv})\ . \end{cases}
$$

The points ω, τ, $A_3, \ldots, A_6$ represent on the hyperquadric Q_4 the six edges of the tetrahedron x, x_u, x_v, x_{uv} associated with the point P_x of the surface S. Adding and subtracting the first and second of equations (61) we obtain

$$
(67)\qquad \begin{cases} \tau_{-1}+\varphi\tau=A_3+A_4\ , \\ \omega_1+\psi\omega=A_3-A_4\ , \end{cases}
$$

and it follows that

$$
(68)\qquad \begin{cases} \omega_1+\tau_{-1}=2A_3-\psi\omega-\varphi\tau\ , \\ \omega_1-\tau_{-1}=-2A_4-\psi\omega+\varphi\tau\ . \end{cases}
$$

Now the point $\omega_1+\tau_{-1}$ is on the hyperquadric Q_4, since it lies in the planar generator $\omega\tau A_3$ which represents the bundle of lines with center at P_x. Therefore *the point $\omega_1+\tau_{-1}$ represents the first directrix of Wilczynski.* Similarly, the point $\omega_1-\tau_{-1}$ lies on the planar generator $\omega\tau A_4$ which represents the ruled tangent plane of S at P_x, and therefore *the point $\omega_1-\tau_{-1}$ represents the second directrix of Wilczynski.* Incidentally, we observe that *the point A_3 represents the projective normal, and therefore the line $A_3(\omega_1+\tau_{-1})$ represents the first canonical pencil, while the line $A_4(\omega_1-\tau_{-1})$ represents the second canonical pencil, at the point P_x of the surface S.*

At this place we exhibit a formula which will be useful later. Let us consider in ordinary space a surface S generated by a point P_x, and let us define two points Y, Z by placing

$$
\begin{aligned} Y&=y_1x+y_2x_u+y_3x_v+y_4x_{uv}\ , \\ Z&=z_1x+z_2x_u+z_3x_v+z_4x_{uv}\ . \end{aligned}
$$

The general coordinates Ω of the line YZ are found by direct calculation to be given by the formula

$$\Omega = \omega_{12}\omega + \omega_{13}\tau + \omega_{14}A_3 + \omega_{23}A_4 + \omega_{42}A_5 + \omega_{34}A_6 \,, \tag{69}$$

where ω, τ are defined by equations (58); the symbols $A_3, \ldots, A_6$ are defined by equations (66); and the local coordinates ω_{ik} of the line YZ are defined by the usual formula $\omega_{ik} = y_i z_k - y_k z_i$. *The formula* (69) *gives the six general coordinates of a point* Ω *on the hyperquadric* Q_4 *as linear combinations of the general coordinates of the vertices of the pyramid* ω, τ, $A_3, \ldots, A_6$, the coefficients ω_{ik} *being the local coordinates of the point* Ω *with reference to this pyramid.*

In order to bring *the quadric of Lie* into consideration again, we make the following remarks. The intersection of the hyperquadric Q_4 by the osculating plane $\omega\omega_1\omega_2$, at a point P_ω of a v-curve of the net N_ω, represents the osculating regulus along the generator xx_u of the ruled surface of u-tangents constructed at the points of the v-curve through a point P_x on the surface S. Similarly, the intersection of Q_4 by the osculating plane $\tau\tau_{-1}\tau_{-2}$, at the corresponding point P_τ of the u-curve of the net N_τ that passes through P_τ, represents the osculating regulus along the generator xx_v of the ruled surface of v-tangents constructed at the points of the u-curve through the point P_x on S. These two planes are reciprocal polars with respect to the hyperquadric Q_4, and the two reguli lie on the quadric of Lie at the point P_x of the surface S.

It was shown in Section 25 that as the point P_x varies on the surface S the quadric of Lie touches its envelope, besides at P_x, in four other points P_{11}, P_{12}, P_{22}, P_{21} which are the vertices of *the tetrahedron of Demoulin* of S at P_x. We shall now discover other properties of the edges of this tetrahedron, making use of their images on the hyperquadric Q_4 in space S_5. On reference to the equation (III, 26) of the quadric of Lie it is easy to verify that any generator of the regulus thereon containing the u-tangent, $x_3 = x_4 = 0$, has the equations

$$x_2 - hx_4 = 0 \,, \qquad 2(hx_3 - x_1) - (\beta\gamma + \theta_{uv})x_4 = 0 \,,$$

in which h is a parameter. Furthermore, the two points

$$(h, 0, 1, 0) \,, \qquad (0, 2h^2, \beta\gamma + \theta_{uv}, 2h)$$

are on this generator, and hence its line coordinates are

$$[h^2, (\beta\gamma + \theta_{uv})/2, h, -h, 0, 1] \,.$$

Therefore the point Ω corresponding to this generator on the hyperquadric Q_4 is given, according to the formula (69), by

$$\Omega=h^2\omega+(\beta\gamma+\theta_{uv})\tau/2+h(A_3-A_4)+A_6\ . \tag{70}$$

Eliminating τ, A_3-A_4, A_6, one is able to express Ω as a linear combination of ω, ω_1, ω_2:

$$2\Omega=[2h^2+2\psi h+(\beta\psi)_v/\beta-\gamma\varphi-2q]\omega+(2h+a_1+\psi)\omega_1+\omega_2\ . \tag{71}$$

The line $\omega_1\omega_2$ meets the hyperquadric Q_4 in two points which represent the two generators of the quadric of Lie which are the directrices of the congruence common to the complexes M_u, L_v. In order that the point Ω may coincide with one of these points it is necessary and sufficient that the coefficient of ω in equation (71) should vanish. Comparing the equation thus obtained with equation (III, 103) we see that* *the edges $P_{11}P_{12}$, $P_{22}P_{21}$ of the tetrahedron of Demoulin are the directrices of the congruence common to the complexes M_u, L_v. Similarly, the edges $P_{12}P_{22}$, $P_{21}P_{11}$ are the directrices of the congruence common to the complexes M_v, L_u.* Since the edges $P_{11}P_{22}$, $P_{12}P_{21}$ intersect all the other edges it follows that *the edges $P_{11}P_{22}$, $P_{12}P_{21}$ are the two lines common to the four complexes L_u, L_v, M_u, M_v.* Incidentally, it follows that *the lines $P_{11}P_{22}$, $P_{12}P_{21}$ intersect the directrices of Wilczynski.*

On the surface S_ω in space S_5 there are ordinarily just three one-parameter families of curves, called *principal curves,* each of which has the property that its osculating space S_3 at any one of its points P_ω lies in the space $S(2, 0)$ of S_ω at P_ω (see § 54, Chap. VII). We proceed to consider these curves briefly. Their differential equation is found by demanding that each of ω, ω', ω'', ω''' may be a linear combination of ω, ω_u, ω_v, ω_{uu}, ω_{vv}. From this demand on ω, ω', ω'' no new conditions result, but from ω''' we find, on making use of the second of equations (62), the desired differential equation,

$$\beta du^3+\gamma dv^3=0\ . \tag{72}$$

Consequently, *the three families of principal curves on the surface S_ω represent*† *the ruled surfaces of u-tangents circumscribed along the curves of Darboux on the surface S.* The intersection of the hyperquadric Q_4 and the space $S(2, 0)$ at a point P_ω of the surface S_ω represents the special complex of lines intersecting the corresponding v-tangent of the surface S; the intersection of Q_4 and the osculating space S_3 at a point of a curve on the surface

* Godeaux, 1927. 6.

† Bompiani, 1926. 7, p. 399.

S_ω represents the osculating linear congruence along the corresponding generator of the ruled surface of u-tangents circumscribed along a curve on the surface S. This congruence, which is determined by four consecutive generators of the ruled surface, consists of the lines intersecting the flecnode tangents of the generator. If the ruled surface is circumscribed along a curve of Darboux, the lines of the osculating linear congruence along a generator intersect the v-tangent that has the same contact point with the surface S as the generator. Hence we have the theorem of Čech (see Ex. 9 of Chap. III):

Each of the two ruled surfaces of asymptotic tangents circumscribed along a curve of Darboux on a surface in space S_3 has the other as a flecnode surface.

We conclude this section with a few remarks concerning *surfaces whose asymptotic curves belong to linear complexes.* If the asymptotic u-curves on a surface S in ordinary space belong to linear complexes, then ω satisfies an equation of the form $\Sigma V\omega = 0$ in which V is a function of v only. This is the equation of the complex M_u whose second image is the point ω_1. Therefore the locus of the point ω_1 is a v-curve, and $H=0$. The first of equations (63) now shows that *a necessary and sufficient condition that the asymptotic u-curves on a surface in ordinary space belong to linear complexes is* (see Ex. 30 of Chap. III)

$$\beta\gamma - (\log \beta)_{uv} = 0 \; . \tag{73}$$

According to Section 28, the u-curves on the surface S_τ now lie in the same hyperplanes as the u-curves on the surface S_ω, and also lie in spaces S_3; hence the ruled surfaces of v-tangents constructed at the points of the u-curves on the surface S belong to the same linear complexes as these u-curves and also belong to linear congruences. Moreover, the locus of the point τ_{-3} is a v-curve, the sequence of Laplace determined by the net N_ω terminating according to the case of Goursat in the negative direction, and according to the case of Laplace in the positive direction. Exercises 16, 17, 18 give additional results connected with this section.

42. Fubini's theory of *W* congruences. The concluding sections of this chapter will be devoted to W congruences, i.e., congruences on whose focal surfaces the asymptotic curves correspond. In this section Fubini's analytic theory of the projective differential geometry of these congruences will be outlined briefly. The reader who desires to see a more extensive exposition of this theory may consult* the treatise by Fubini and Čech. In the next section parts of this theory will be transformed by the correspond-

* Fubini and Čech, 1926. 1, p. 243.

ence between lines in ordinary space and points on the hyperquadric Q_4 in space S_5.

The first problem is to determine the second focal surface of the congruence of tangents of a one-parameter family of curves on a surface. Let us consider in ordinary space S_3 an integral surface S_x of equations (III, 6) *without the restriction* $\theta = \log \beta\gamma$ characteristic of Fubini's canonical form of the differential equations. On S_x let us consider a curve of the family defined by the equation

$$Adv - Bdu = 0\ ,$$

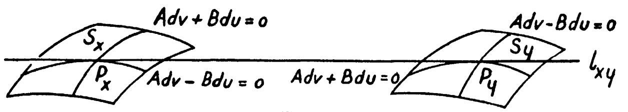

FIG. 32

wherein the coefficients A, B are functions of u, v (see Fig. 32). Any point P_y on the tangent l_{xy} at a point P_x of this curve is defined by

$$y = \mu x + 2(Ax_u + Bx_v) \qquad (\mu \text{ scalar})\ . \tag{74}$$

As the point P_x varies on a curve of the conjugate family

$$Adv + Bdu = 0\ ,$$

the line l_{xy} describes a developable surface. The point P_y is the focal point of l_{xy} regarded as a generator of this developable, if, and only if, μ is given by the formula

$$\mu = -A_u - B_v - A\theta_u - B\theta_v + A(B_u + \beta A)/B + B(A_v + \gamma B)/A. \tag{75}$$

When u, v vary independently, the locus of the point P_y is a surface S_y and the line l_{xy} generates a congruence xy. One focal surface of this congruence is S_x, and when μ is given by the formula (75) the other focal surface is S_y.

In order to find a condition necessary and sufficient that the congruence xy may be a W congruence, let us first demand that the u-curves on the focal surface S_y shall be asymptotic curves. Then y must satisfy the equation $(y_{uu}, y, y_u, y_v) = 0$. By means of equations (74), (75) this equation can be reduced to

$$[(A_v + \gamma B)/A]_u = [(B_u + \beta A)/B]_v\ ; \tag{76}$$

the details of the reduction will be omitted. The symmetry of equation (76) shows that *the v-curves on the surface S_y are asymptotic if the u-curves are.* In

fact, *equation* (76) *is a necessary and sufficient condition that the congruence* xy *may be a* W *congruence.*

Equation (76) can be simplified. In the first place this equation shows that there exists a function φ defined, except for a constant factor, by the differential equations

$$(\log \varphi)_v = (A_v + \gamma B)/A \ , \qquad (\log \varphi)_u = (B_u + \beta A)/B \ .$$

If two functions $\bar{A}$, $\bar{B}$ are defined by placing $A = \varphi\bar{A}$, $B = \varphi\bar{B}$, these definitions can be used to eliminate φ from its two defining differential equations. Thus we obtain

$$\bar{A}_v = -\gamma\bar{B} \ , \qquad \bar{B}_u = -\beta\bar{A} \ .$$

If now the dashes are dropped, we have

$$A_v = -\gamma B \ , \qquad B_u = -\beta A \ . \tag{77}$$

These equations are equivalent to equation (76) under the choice of the proportionality factor φ, which has no geometrical significance. It is on these equations that Fubini's theory of W congruences rests.

The next step is to calculate a system of differential equations for the surface S_y. Let us rewrite equation (75), taking advantage of (77), and also let us define two functions λ, N in the following equations:

$$\left\{\begin{array}{l} \mu = -A_u - B_v - A\theta_u - B\theta_v \ , \qquad \lambda = -A_u + B_v - A\theta_u + B\theta_v \ , \\ N = \lambda\mu + 2A(\mu_u + 2Ap) - 2B(\mu_v + 2Bq) \ . \end{array}\right. \tag{78}$$

Then actual calculation gives

$$\left\{\begin{array}{l} y_u = (\mu_u + 2Ap)x - \lambda x_u + 2Bx_{uv} \ , \\ y_v = (\mu_v + 2Bq)x + \lambda x_v + 2Ax_{uv} \ , \\ Nx = \lambda y + 2Ay_u - 2By_v \ , \\ y_{uu} - \theta_u y_u + \beta y_v - \pi y = N_u x/2A \ , \\ y_{vv} + \gamma y_u - \theta_v y_v - \chi y = -N_v x/2B \ , \end{array}\right. \tag{79}$$

where π, χ now have the values

$$\pi = p + \beta_v + \beta\theta_v \ , \qquad \chi = q + \gamma_u + \gamma\theta_u \ .$$

Eliminating x from the last three of equations (79) we reach *a system of differential equations for the surface* S_y. This system is of the same form as

the system we are using for the surface S_x; its coefficients, indicated by dashes, are given by the following formulas:

$$(80)\quad \begin{cases} \bar{p}=\pi+\lambda N_u/2AN\,, & \bar{\theta}_u=\theta_u+(\log N)_u\,, & \bar{\beta}=-\beta-BN_u/AN\,, \\ \bar{q}=\chi-\lambda N_v/2BN\,, & \bar{\gamma}=-\gamma-AN_v/BN\,, & \bar{\theta}_v=\theta_v+(\log N)_v\,. \end{cases}$$

If two functions S, T are defined by placing

$$(81)\qquad N_u=2AS\,,\qquad N_v=-2BT\,,$$

then it can be shown that S, T satisfy the equations

$$(82)\qquad S_v=\beta T\,,\qquad T_u=\gamma S\,,$$

whence it follows at once that *the function N satisfies the equation of Laplace,*

$$(83)\qquad N_{uv}+\gamma BN_u/A+\beta AN_v/B=0\,.$$

The geometrical significance of the condition $N=\text{const.}$ can readily be found. It will be observed, on inspection of the third of equations (79), that we are only interested in the case $N\neq 0$, since the locus S_y is not a proper surface if $N=0$. Moreover, equations (80) show that if $N=\text{const.}$, then system (80) reduces to the system given in Exercise 15 of Chapter III for the coordinates of the tangent plane of the surface S_x. In this case the tangent planes of the surface S_x and the points P_y are related by a correlation in which corresponding point and plane are in united position, and which is therefore a null system. Consequently we have the theorem:

A W congruence with $N=\text{const.}$ belongs to a linear complex.

Let us consider a W congruence of which one focal surface is ruled and one not ruled. If the surface S_y is ruled with the u-curves for generators, then $\bar{\beta}=0$ and hence $\beta=-BN_u/AN$. Equations (77), (83) may be used to show that

$$(84)\qquad (\log B/N)_u=0\,,\qquad (\log B/\beta)_v=0\,,\qquad (\log N)_{uv}=\beta\gamma\,;$$

from these equations it follows that equation (73) is satisfied. Thus we reach the following theorem.

If a non-ruled surface S_x is one focal surface of a W congruence of which the other focal surface S_y is ruled, then the asymptotic curves on S_x that correspond to the generators on S_y belong to linear complexes.

Let us finally consider a W congruence of which one focal surface is a

quadric. If the surface S_x is a quadric then $\beta=\gamma=0$ and by Exercise 4 of Chapter III it is possible to make $p=q=0$, $\theta=$ const. Then we find

$$(85)\qquad \begin{cases} A=U\,, \quad B=V\,, \quad \mu=-U'-V'\,, \quad \lambda=-U'+V'\,, \\ N=U'^2-V'^2-2UU''+2VV''\,, \\ \bar\beta=2VU'''/N\,, \quad \bar\gamma=2UV'''/N\,, \end{cases}$$

where U, V are functions of u alone and of v alone respectively and accents indicate differentiation. Therefore S_y is a quadric surface in case U and V are polynomials at most of the second degree in u and v respectively with constant coefficients; in this case the congruence xy belongs to a linear complex, since $N=$ const. If the surface S_y is ruled but not a quadric, we may suppose $\bar\gamma=0$, $\bar\beta\neq0$; still the ruled surface S_y belongs to a linear congruence, since the equation $\bar\beta=0$ has two solutions $v=$ const. Finally, even if the surface S_y is not ruled, all of the asymptotic curves on the surface S_y belong to linear complexes, since

$$(\log\bar\beta)_{uv}=(\log\bar\gamma)_{uv}=\bar\beta\bar\gamma\,.$$

The reader is referred to Exercises 19, 20 for additional results.

43. *W* congruences in space R_3. The correspondence previously studied between lines of ordinary ruled space R_3 and points on a hyperquadric Q_4 in space S_5 has been used* by Terracini to connect Fubini's theory of W congruences with the theory of conjugate nets, and in particular with the transformation of Ribaucour. Some of these connections will be explained in this section, in which the exposition will follow in its salient features the memoir of Terracini just cited.

Darboux proved† that *the line coordinates of a generator of a W congruence satisfy a linear partial differential equation of the second order. When the parameters are suitably chosen this is an equation of Laplace.* We wish to compute this equation for the W congruence xy of the preceding section. For this purpose let us choose the proportionality factor for the coordinates t of a generator l_{xy} of this congruence so that

$$(86)\qquad t=A\omega+B\tau\,,$$

it being understood that the coefficients A, B satisfy equations (77) and ω, τ are defined by equations (58). Then, by means of equations (59), (60), (77), differentiation and elimination lead to the following *equation of Laplace satisfied by t,*

$$(87)\qquad t_{uv}=[\beta\gamma-(\log A)_u\,(\log B)_v]t+(\log B)_v t_u+(\log A)_u t_v\,.$$

* Terracini, 1927. 7. † Darboux, 1889. 1, p. 345.

For equation (87) the Laplace-Darboux invariants H, K are given by

$$H=\beta\gamma-(\log B)_{uv}\ , \qquad K=\beta\gamma-(\log A)_{uv}\ . \tag{88}$$

It follows at once that $H=K$ *if, and only if, the developables of the W congruence xy touch the surface S_x along the curves of an isothermally conjugate net.* In this case, making $A=B$ by a transformation of parameters, we obtain from (77) the condition $\beta_v=\gamma_u$; thus we find that *S_x is an R surface*, i.e., sustains an R net (see Ex. 22).

If we define the coordinates $\bar{\omega}$, $\bar{\tau}$ of the u-tangent and the v-tangent respectively at a point P_y of the surface S_y by the formulas

$$\bar{\omega}=e^{-\bar{\theta}}(y,\,y_u)\ , \qquad \bar{\tau}=e^{-\bar{\theta}}(y,\,y_v)\ , \tag{89}$$

where the function $\bar{\theta}$ is defined, except for an additive constant, by

$$\bar{\theta}=\theta+\log N\ ,$$

then $\bar{\omega}$, $\bar{\tau}$ satisfy equations of the same form as those satisfied by ω, τ but involving the dashed coefficients given by equations (80) in place of the coefficients of equations (III, 6). Moreover, it is not difficult to establish the linear relation

$$A\omega+B\tau=-A\bar{\omega}+B\bar{\tau}\ . \tag{90}$$

It was shown at the conclusion of Section 39 that when a focal net of a quadratic congruence is subjected to a transformation of Ribaucour, this transformation carries the congruence into another quadratic congruence. It was also shown that two corresponding lines of these congruences intersect in a point which generates a quadratic net conjugate to both congruences. This theory can be applied to the quadratic congruences $\omega\tau$ and $\bar{\omega}\bar{\tau}$ which represent on the hyperquadric Q_4 the focal surfaces S_x and S_y of a W congruence. In fact, we shall show in the next paragraph that the two nets N_ω and $N_{\bar{\omega}}$ on the hyperquadric Q_4 are in the relation of a transformation of Ribaucour, and that so are the nets N_τ and $N_{\bar{\tau}}$. Two corresponding lines of the congruences $\omega\tau$ and $\bar{\omega}\bar{\tau}$ are shown by equation (86) and the linear relation (90) to intersect in the point t that represents the line l_{xy}. This point generates a net N_t on Q_4, whose Laplace equation is (87), and which represents the W congruence xy.

Referring to Figure 33, we proceed to prove that *the nets N_ω and $N_{\bar{\omega}}$ are in*

the relation of a transformation of Ribaucour. In the latter part of Section 32 we solved the problem of finding all congruences conjugate to a given net.

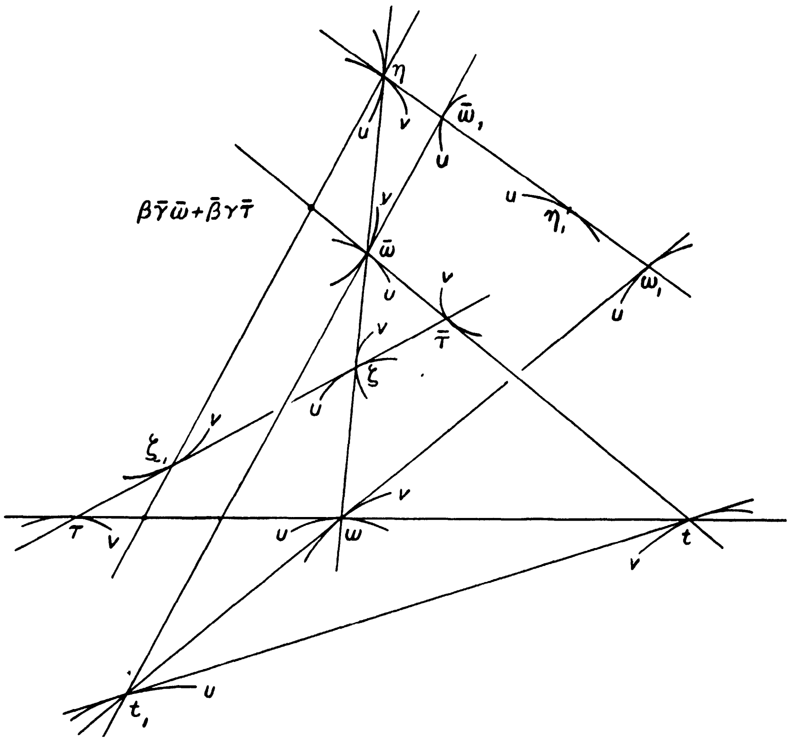

FIG. 33

Applying the theory there developed to the net N_ω we write the adjoint of the first of equations (60) in the form

$$\varphi_{uv}=[\beta\gamma-(\log\beta)_{uv}]\varphi-(\log\beta)_v\varphi_u\,. \tag{91}$$

The focal points η, ζ of a generator of the most general congruence conjugate to the net N_ω are determined from the equations

$$\begin{cases}\eta_u=\varphi_u\omega\,, & \eta_v=\varphi[\omega_v-(\log\beta)_v\omega]\,,\\ \zeta\ =\eta-\varphi\omega\,, & \end{cases} \tag{92}$$

where the function φ is a solution of equation (91). Now it happens that S/β is a particular solution of equation (91). With this solution equations (92) give

$$\eta = N(\omega+\bar{\omega})/2B + S\omega/\beta\ , \qquad \zeta = N(\omega+\bar{\omega})/2B\ . \tag{93}$$

Since the line $\eta\omega$ passes through the point $\bar{\omega}$, the nets N_ω, $N_{\bar{\omega}}$ are both conjugate to the congruence $\eta\zeta$ and hence are in the relation of a transformation of Ribaucour, as was to be proved. The proof for the nets N_τ, $N_{\bar{\tau}}$ can be made similarly; the first and second focal points, analogous to the points η, ζ are the points η', ζ' given by

$$\eta' = N(\bar{\tau}-\tau)/2A\ , \qquad \zeta' = N(\bar{\tau}-\tau)/2A + T\tau/\gamma\ ; \tag{94}$$

the solution of the adjoint of the second of equations (60) used is T/γ. It will be observed that the points ζ, η' coincide in virtue of the relation (90), and that the points ζ', ζ_{-1} coincide.

We shall next show how to prove that *the net N_t representing the W congruence xy is the most general net conjugate to the congruence $\omega\tau$*. The problem of determining all nets conjugate to a given congruence was solved in Section 32. Using equations (59) in place of (IV, 69) the reader may prove that equations (IV, 71), (IV, 72), (IV, 74) give for the most general net conjugate to the congruence $\omega\tau$ precisely the net N_t of equation (87).

Another application of the theory in Section 32 for the determination of all congruences conjugate to a given net will show that *the congruence $\bar{\omega}\bar{\tau}$ is conjugate to the net N_t*. The adjoint of equation (87) is

$$\varphi_{uv} = [\beta\gamma - (\log A)_u\,(\log B)_v - (\log AB)_{uv}]\varphi - (\log B)_v\varphi_u - (\log A)_u\varphi_v\ . \tag{95}$$

The focal points η'', ζ'' of a generator of the most general congruence conjugate to the net N_t are determined from the equations

$$\left\{\begin{array}{ll} \eta''_u = [\varphi_u + (\log A)_u\varphi]t\ , & \eta''_v = \varphi[t_u - (\log B)_v t]\ , \\ \zeta'' = \eta - \varphi t\ , & \end{array}\right. \tag{96}$$

where the function φ is a solution of equation (95). One solution of equation (95) is N/AB; with this solution equations (96) give

$$B\eta'' = -N\bar{\omega}\ , \qquad B\zeta'' = -N(\bar{\omega} + t/A)\ . \tag{97}$$

The demonstration is completed by observing that this line $\eta''\zeta''$ coincides with the line $\bar{\omega}\bar{\tau}$. Incidentally, another solution of equation (95) is $1/AB$; with this solution equations (96) give

$$(98) \qquad B\eta''=\omega\ , \qquad B\zeta''=\omega-t/A\ .$$

This line $\eta''\zeta''$ coincides with the line $\omega\tau$; consequently the congruence thus determined as conjugate to the net N_t is the congruence $\omega\tau$.

EXERCISES

1. If the product of two transformations F is a transformation F the three nets involved either are conjugate to the same congruence, or else are harmonic to the same congruence. (In the first case the three nets are said to form *a conjugate triad*, and in the second, *a harmonic triad.*) SLOTNICK, 1928. 3, p. 202

2. Prove *the theorem of permutability* for transformations F, namely, that if two nets N_1, N_2 are both F transforms of a net N, then there exist ∞^2 nets N' such that each N' is an F transform of both N_1 and N_2. Prove that four corresponding points on four nets N_1, N_2, N, N' lie in a plane which envelops a net, and that the tangent planes at four corresponding points of the nets meet in a point which generates a net.
EISENHART, 1923. 3, p. 45

3. Using the points x, x_u, x_v, y in the notation of system (16) as the vertices of a local tetrahedron of reference, show that the equation of the quadric having contact of the second order with the surface S_x at a point P_x and contact of the first order with S_y at the corresponding point P_y is

$$Lx_2^2+Nx_3^2-2x_1x_4=0\ .$$

Show also that the equation of the quadric defined with the rôles of x and y interchanged is

$$npx_2^2+mqx_3^2-2mnx_1x_4=0\ .$$

Discuss the cones contained in the pencil determined by these quadrics.
LANE, 1929. 1, p. 466

4. In the coordinate system of Exercise 3, the equations of the ray of the point P_x are

$$x_4=x_1+bx_2+ax_3=0\ ,$$

and those of the ray of P_y are

$$x_1=bmx_2+anx_3+mnx_4=0\ .$$

Hence the cross ratio of the second and first focal points x_u, x_v of the generator of the harmonic congruence and the points ρ, σ where this line meets the rays of P_x

and P_y is the conjugate invariant R. Dually, the equation of the plane containing the line xy and the axis of the point P_x is

$$\beta N x_2 - \gamma L x_3 = 0 ,$$

and the equation of the plane containing this line and the axis of P_y is

$$q\beta x_2 - p\gamma x_3 = 0 .$$

Hence the cross ratio of the second and first focal planes ($x_3=0$ and $x_2=0$) of the line xy and these two planes is the harmonic invariant S.

LANE, 1929. 1, p. 464

5. If a net N_x with equal invariants is conjugate to a congruence with focal nets N_η, N_ζ, the point which is the harmonic conjugate of a point P_x with respect to the corresponding points P_η, P_ζ generates a net with equal invariants which is conjugate to the congruence.

6. If a net N_x in space S_n has equal invariants there exists a conjugate congruence with focal nets N_η, N_ζ such that the net generated by the point which is the harmonic conjugate of P_x with respect to P_η, P_ζ is in a hyperplane S_{n-1}.

7. The conic of Koenigs for a point of a net N_x with equal invariants and the conic of Koenigs for the corresponding point of a net N_y in the relation of a transformation of Koenigs to N_x intersect the corresponding generator of the harmonic congruence of the transformation in the same two points.

8. The quadrics of the pencil (26) intersect the line xy in pairs of points in an involution whose double points are the focal points P_η, P_ζ; the quadrics of the pencil that are tangent to the line xy are the cones of the pencil, the points of contact being the focal points P_η, P_ζ.

9. Find the vertices of the cones in the pencil (26). If the two conics determining this pencil are the conics of Koenigs, according to Exercise 7, the vertices are the points η_1, ζ_{-1} given by equations (14).

10. In the situation discussed in Section 35 any line through a point P_x of a net N_x and lying in the tangent plane of N_x at P_x intersects the conics of Koenigs (21) in four harmonic points if, and only if, $H+K=0$.

11. The cross ratio of the four points x_{-1}, ζ, y_{-1}, ζ_{-1} of Section 34 is $-bn_v/nK$, and that of the points x_1, η, y_1, η_1 is $-am_u/mH$.

12. In the situation of Section 37 the differential equations of the asymptotic curves on the surfaces S_x, S_y are respectively

$$L du^2 + 2M\, du dv + N dv^2 = 0 , \qquad L' du^2 + 2M' du dv + N' dv^2 = 0 .$$

Use equation (40) to prove that the developables of the congruence $\rho\sigma$ correspond to a conjugate net on S_x (supposed non-developable) if, and only if, the develop-

ables of the congruence xy (supposed determinate and given by $dudv=0$) intersect the surface S_y in a conjugate net ($M'=0$). State the result when x and y are interchanged.
GROVE, 1928. 2, p. 487

13. Necessary and sufficient conditions that the developables of the congruence $\rho\sigma$ be determinate and correspond to the developables of the congruence xy are found from equation (40) to be

$$mLM'-nML'=mMN'-nNM'=0\ , \qquad m(LN'+MM')-n(NL'+MM')\neq 0\ .$$

These conditions are satisfied if the developables of the congruence xy cut both surfaces S_x, S_y in conjugate nets ($M=M'=0$), or else if these developables cut both S_x, S_y in their asymptotic curves ($L=N=L'=N'=0$) . Ruling out these two cases prove that

$$LN/M^2=L'N'/M'^2\ .$$

Hence prove that at a point P_x the cross ratio of the two tangents of the curves $dudv=0$, in which the developables of the congruence xy intersect S_x, and the two asymptotic tangents is equal to the cross ratio (in the proper order) of the two tangents at the corresponding point P_y of the curves $dudv=0$, in which the developables of the congruence xy intersect S_y, and the two asymptotic tangents at P_y.
GROVE, 1928. 2, p. 487

14. Necessary and sufficient conditions that the foci of each line $\rho\sigma$ be determinate and lie on the parametric tangents at the corresponding point P_x are found from equation (41) to be

$$MN'-NM'=ML'-LM'=0\ , \qquad m(LN'-MM')-n(NL'-MM')\neq 0\ .$$

These conditions are satisfied if the developables of the congruence xy cut both surfaces S_x, S_y in conjugate nets ($M=M'=0$), or else in their asymptotic curves ($L=N=L'=N'=0$), or else if the asymptotics correspond on the surfaces S_x, S_y (($L/L'=M/M'=N/N'$). Ruling out these three cases prove that

$$LN/M^2=L'N'/M'^2\ ,$$

so that the cross ratio property of Exercise 13 is present.
GROVE, 1928. 2, p. 487

15. Every quadratic net with equal invariants in ordinary space is an integral net of a system of equations which can be written in the form

$$x_{uv}=cx\ , \qquad x_{uu}+x_{vv}=dx\ ,$$

where the coefficients c, d are defined by

$$c=\varphi(u+v)-\psi(u-v)\ , \qquad d=2[\varphi(u+v)+\psi(u-v)]\ ,$$

the functions φ, ψ being arbitrary functions of the arguments indicated.
TZITZÉICA, 1924. 3, Chap. IV

16. The planes $\omega_1\omega_2\omega_3$ and $\tau_{-1}\tau_{-2}\tau_{-3}$ in the situation of Section 41 are reciprocal polars with respect to the hyperquadric Q_4, and therefore intersect Q_4 in conics representing reguli on a quadric surface. This quadric contains the four edges of the tetrahedron of Demoulin that lie on the quadric of Lie at a point x of the surface S; hence the two quadrics are tangent at the vertices of this tetrahedron.

GODEAUX, 1927. 6

17. For each point ω on the surface S_ω of Section 41, the point

$$h(h+k_3)\omega-(hk_2+k_4)\tau+(h+k_3)A_3+hk_1A_4-k_1A_6$$

lies on the hyperquadric Q_4 and represents a generator of the regulus containing the u-tangent on the quadric (III, 86) at a point x on the surface S. As the generator varies over the regulus, the locus of this point is the conic of intersection of Q_4 and the plane determined by the three points

$$\omega\,, \qquad k_3\omega-k_2\tau+A_3+k_1A_4\,, \qquad k_4\tau-k_3A_3+k_1A_6\,.$$

This plane lies in the space $S(2, 0)$ at the point ω of the surface S_ω. The regulus containing the u-tangent on the asymptotic osculating quadric (III, 29) of a curve at the point x on the surface S corresponds to the intersection of the hyperquadric Q_4 and the osculating plane at the point ω of a curve on S_ω. For a planar system of curves as defined in Section 22, these two planes intersect in a straight line.

BOMPIANI, 1926. 2

18. The osculating plane at a point ω of a principal curve on the surface S_ω and the osculating plane at this point of the harmonic reflection of this principal curve in the parametric curves, which represents the ruled surface of u-tangents touching the surface S along a curve of Segre, intersect in a straight line. The three principal curves at the point ω thus determine three straight lines which lie in the plane of the three points

$$\omega\,, \qquad 3\omega_{uu}-(\log\beta/\gamma)_u\omega_u\,, \qquad 3\omega_{vv}-(\log\beta^5\gamma)_v\omega_v\,.$$

19. If both focal surfaces S_x, S_y of a W congruence are ruled, and if to the curved asymptotics on S_x correspond generators on S_y, then S_y is a quadric and the generators of the second family on S_y correspond to the generators on S_x.

SEGRE, 1913. 1

20. If the curves of Darboux correspond on the two focal surfaces of a congruence, the congruence is a W congruence and both surfaces have the property of being *isothermally asymptotic.* (For an integral surface of equations (III, 6) this means that $(\log\beta/\gamma)_{uv}=0$.) Then it is possible to make $\beta=\gamma$ by a transformation of parameters.

FUBINI and ČECH, 1926. 1, p. 283

21. The cross ratio of the points ω, $\bar\omega$ and the focal points η, ζ given by equations (93) is $-\beta/\bar\beta$; the cross ratio of the points τ, $\bar\tau$ and the focal points η', ζ' given by

equations (94) is $-\gamma/\bar{\bar{\gamma}}$ (see Ex. 11). Therefore the harmonicness of these two sets of points implies that the curves of Darboux correspond on the two focal surfaces S_x, S_y of the W congruence xy in space S_3.

22. If in Section 42 the congruence xy consists of the tangents to the curves $dv+\lambda du=0$ on the surface S_x, so that $B/A=-\lambda$, the condition that the congruence be a W congruence becomes

$$(\log \lambda)_{uv}+(\gamma\lambda)_u-(\beta/\lambda)_v=0 .$$

If also the tangents to the conjugate curves $dv-\lambda du=0$ form a W congruence, the conjugate net $dv^2-\lambda^2du^2=0$ is isothermally conjugate and the net is an R net. Then by a transformation of parameters it is possible to make $\lambda=1$ and $\beta_v=\gamma_u$ (see Ex. 15 of Chap. IV).

23. For the point ζ given by the second of equations (93) prove that

$$\zeta_u=-S\tau , \qquad \zeta_v=-T\omega ,$$

and hence derive the equation

$$\zeta_{uv}=\beta T\zeta_u/S+\gamma S\zeta_v/T .$$

Of this equation a particular solution is the function N defined in equations (78). The Levy transform of the net N_ζ in the v-direction by means of this solution is the net $N_{\bar{\omega}}$, while the Levy transform of the net N_ζ in the same direction by means of a constant solution is the net N_ω. TERRACINI, 1927. 7

24. The point ζ in Exercise 23 is the pole, with respect to the hyperquadric Q_4, of the space S_4 whose intersection with Q_4 represents the osculating linear complex along the generator l_{xy} of the W congruence xy in space S_3. TERRACINI, 1927. 7

25. In the notation of Exercises 3, 4 the lines xx_u, yx_v, $\eta\rho$, $\zeta\sigma$ lie on the quadric

$$ax_1x_3-bmx_2x_4=0 ;$$

and the lines xx_v, yx_u, $\eta\sigma$, $\zeta\rho$ lie on the quadric

$$bx_1x_2-anx_3x_4=0 .$$

LANE, 1929. 1, p. 470. This result is due to Mendel.

26. In the situation discussed in Section 37 consider any point P not a focal point on any line l_{xy} of the congruence xy. Construct the two focal planes of l_{xy} and the two tangent planes at P of the two ruled surfaces of the congruence that contain l_{xy} and intersect the surfaces S_x, S_y in parametric curves thereon. Prove that the conditions $m=n$, $st\neq 0$ are necessary and sufficient that the first two planes separate the second two harmonically, whatever be the position of P on l_{xy}. (In

this case the parametric net of ruled surfaces of the congruence, composed of the ruled surfaces for which $u=\text{const.}$ and those for which $v=\text{const.}$, is said to be *a conjugate net of ruled surfaces.*) WILCZYNSKI, 1920. 3, p. 203

27. Calculate the integrability conditions of equations (42). Determine the transformations that leave the parametric net invariant, and calculate the effect of this transformation on the coefficients of the equations. GREEN, 1920. 2

28. Consider a net N_x with x satisfying equation (1) and consider a different net N_y with y likewise satisfying (1). If the point z defined by an expression of the form $z=x+hy$ describes a net when u, v vary, so that z also satisfies a Laplace equation with independent variables u, v, then this equation is identical with (1) and $h=\text{const.}$ The surfaces sustaining four such nets N_z intersect a line xy in four points whose cross ratio is the same for all lines xy.

CHAPTER VI

METRIC AND AFFINE APPLICATIONS

Introduction. The aim of this chapter is to connect the projective differential geometry of surfaces, as developed in the earlier chapters of this book, with the metric and affine differential geometries of surfaces in ordinary space. It is well known that the group of projective transformations in ordinary space contains as a subgroup the group of rigid motions. Therefore every invariant under the group of projections is also an invariant under the group of motions. The projective geometry of a configuration is the study of the properties of the configuration that are invariant under the projective group, and the metric geometry of the configuration is the study of the properties of the configuration that are invariant under the group of motions, which may be called the metric group. Therefore all of projective geometry may properly be included in metric geometry; every projective theorem is also a metric theorem, and every projectively defined configuration has metric properties. It is a matter of history that many theorems of a projective nature were discovered and included in metric geometry before projective geometry was organized as a separate science. The metric investigation of configurations that have appeared first in projective geometry has been very fruitful as a general method of research, and is to be thought of in contrast with a second method, namely, the attempt to find projective analogues of familiar configurations and theorems of a metric nature.

The general projective point coordinate system which is used in analytic projective geometry of ordinary space can be specialized so as to become an ordinary orthogonal cartesian coordinate system with equal units on all three axes. This specialization is accomplished by projecting one face of the tetrahedron of reference into the plane at infinity, by projecting the other three faces into three mutually orthogonal planes, and by suitably disposing of the unit point. Except in investigations involving elements at infinity it is customary to use *non-homogeneous* coordinates in metric analytic geometry.

A projective transformation that makes finite points correspond to finite points and makes points at infinity correspond to points at infinity is called an *affine transformation*. The group of affine transformations is therefore a subgroup of the projective group; moreover, the affine group contains the metric group as a subgroup.

The contents of this chapter may be outlined as follows. In the opening section a few of the fundamental facts and formulas of the metric differential geometry of surfaces in ordinary space are collected for convenience of reference. In Section 45 some of the elementary portions of the geometry of the sphere are reviewed and pentaspherical coordinates are introduced to prepare the way for the definition and a preliminary account, in the following section, of a certain correspondence between points in ordinary metric space and points on a hyperquadric in a projective space S_4. This correspondence is applied in Section 47 in studying the lines of curvature on a surface in ordinary space.

In Section 48 a local trihedron at a point of a surface is used as the basis of a local cartesian coordinate system for the purpose of investigating metric properties of projectively defined configurations associated with the point. The connection between Fubini's normal coordinates, which were employed so extensively in Chapter III, and cartesian orthogonal coordinates is established in Section 49.

Only the last two sections are concerned with affine geometry, and even that is of a very restricted kind. Section 50 is devoted to the elements of the special kind of affine geometry of surfaces which leaves the origin invariant. Finally in Section 51 a certain projectively defined class of surfaces, namely, *surfaces with indeterminate directrix curves*, is studied, first from the projective point of view and then from the point of view of the special affine geometry.

44. Metric geometry of surfaces. It will be convenient for later use to collect here some of the definitions, formulas, and theorems of the metric differential geometry of surfaces in ordinary space. The notation employed will not diverge far from that of the treatise by Bianchi and of Eisenhart's *Differential Geometry*. These works may be consulted* by any one wishing to read a more extended account of the subject than the rather sketchy summary offered here.

Let us establish a left-handed orthogonal cartesian coordinate system with equal units on all three axes in ordinary metric space. If the coordinates x, y, z of a point P in this space are single-valued analytic functions of two independent variables u, v, such that not all of the jacobians A, B, C defined by the formulas

$$(1) \qquad A=y_uz_v-y_vz_u\ , \qquad B=z_ux_v-z_vx_u\ , \qquad C=x_uy_v-x_vy_u$$

* Eisenhart, 1909. 1; Bianchi, 1922. 5.

are zero, the locus of the point P, as u, v vary, is by definition *a proper analytic surface* S. *The tangent plane* at the point P of the surface S has the equation

$$A(\xi - x) + B(\eta - y) + C(\zeta - z) = 0 , \tag{2}$$

in current coordinates ξ, η, ζ.

The square of the element of arc of a curve on a surface is given by

$$ds^2 = dx^2 + dy^2 + dz^2 = Edu^2 + 2Fdudv + Gdv^2 , \tag{3}$$

where E, F, G are defined by the formulas

$$E = \Sigma x_u^2 , \qquad F = \Sigma x_u x_v , \qquad G = \Sigma x_v^2 , \tag{4}$$

the summation being for cyclical permutation of x, y, z. The binary quadratic differential form in the last member of (3) is called *the first fundamental form*, and E, F, G are spoken of as *the first fundamental coefficients*, of the surface. Denoting by H^2 the discriminant $EG - F^2$ of this form we have the equalities

$$H^2 = EG - F^2 = A^2 + B^2 + C^2 . \tag{5}$$

Then, if X, Y, Z denote *the direction cosines of the normal* at a point of the surface, we find from equation (2) the formulas

$$X = A/H , \qquad Y = B/H , \qquad Z = C/H . \tag{6}$$

The equation of *the minimal curves*, or curves of length zero, on the surface is

$$Edu^2 + 2Fdudv + Gdv^2 = 0 . \tag{7}$$

If two surfaces have their points in one-to-one correspondence so that the length of any curve on one surface, calculated between any two points on it, is equal to the length of the corresponding curve between the corresponding points on the other surface, the surfaces are said to be *applicable* and the correspondence is called an *applicability*. If the parameters on two surfaces with their points in correspondence are chosen so that corresponding points have the same curvilinear coordinates, it can be shown that *a necessary and sufficient condition that the correspondence be an applicability is that the first fundamental coefficients of one surface be respectively equal to those of the other surface*. The surfaces then have the same first fundamental form.

The equation of *the asymptotic curves* on a surface is

$$Ddu^2+2D'dudv+D''dv^2=0\ , \tag{8}$$

where the coefficients D, D', D'' are defined by placing

$$D=\Sigma X x_{uu}\ ,\qquad D'=\Sigma X x_{uv}\ ,\qquad D''=\Sigma X x_{vv}\ . \tag{9}$$

The binary quadratic differential form in equation (8) is called *the second fundamental form*, and its coefficients are spoken of as *the second fundamental coefficients*, of the surface.

The Christoffel symbols $\{ik,\ l\}$ *of the second kind for the first fundamental form of a surface* may be defined by the formulas

$$\left\{\begin{array}{ll} 2H^2\{11,\ 1\}=GE_u+FE_v-2FF_u\ , & 2H^2\{22,\ 2\}=EG_v+FG_u-2FF_v\ , \\ 2H^2\{21,\ 1\}=GE_v-FG_u\ , & 2H^2\{12,\ 2\}=EG_u-FE_v\ , \\ 2H^2\{22,\ 1\}=-FG_v-GG_u+2GF_v\ , & 2H^2\{11,\ 2\}=-FE_u-EE_v+2EF_u \end{array}\right. \tag{10}$$

The differential equations

$$\left\{\begin{array}{l} x_{uu}=\{11,\ 1\}x_u+\{11,\ 2\}x_v+DX\ , \\ x_{uv}=\{21,\ 1\}x_u+\{12,\ 2\}x_v+D'X\ , \\ x_{vv}=\{22,\ 1\}x_u+\{22,\ 2\}x_v+D''X\ , \\ X_u=(FD'-GD)x_u/H^2+(FD-ED')x_v/H^2\ , \\ X_v=(FD''-GD')x_u/H^2+(FD'-ED'')x_v/H^2 \end{array}\right. \tag{11}$$

are shown in treatises on metric differential geometry to be satisfied not only by x, X but also by y, Y and z, Z. These are *the fundamental differential equations in the metric differential geometry of surfaces in ordinary space.*

There are *three integrability conditions* of equations (11). One of these is *the equation of Gauss,*

$$HK=(H\{11,\ 2\}/E)_v-(H\{12,\ 2\}/E)_u\ , \tag{12}$$

where K is the total curvature which will be defined in the first of equations (15). The other two are *the equations of Codazzi,*

$$\left\{\begin{array}{l} D_v-D'_u-\{21,\ 1\}D+(\{11,\ 1\}-\{12,\ 2\})D'+\{11,\ 2\}D''=0\ , \\ D''_u-D'_v-\{12,\ 2\}D''+(\{22,\ 2\}-\{21,\ 1\})D'+\{22,\ 1\}D=0\ . \end{array}\right. \tag{13}$$

The lines of curvature on a surface may be defined analytically by setting equal to zero the jacobian of the first and second fundamental forms of the surface. Then the curvilinear differential equation of the lines of curvature on a surface is

$$(ED'-FD)du^2+(ED''-GD)dudv+(FD''-GD')dv^2=0\,. \tag{14}$$

The lines of curvature thus defined analytically have the following characteristic geometric property. At each point P of a surface S the curve of section of S by a plane through the normal has a radius of curvature which is a minimum if the plane contains the tangent of one line of curvature through P, and a maximum if it contains the other. These extreme radii are called *principal radii of normal curvature* and are denoted in this book by R_1, R_2.

The total curvature K and the mean curvature K_m at a point of a surface may be defined by the formulas

$$\begin{cases} K\ =1/R_1R_2=(DD''-D'^2)/H^2\,, \\ K_m=1/R_1+1/R_2=(ED''+GD-2FD')/H^2\,. \end{cases} \tag{15}$$

If $K=0$ at every point of a surface, the surface can be proved to be a *developable*. On the other hand, if $K_m=0$ at every point of a surface, the surface is called a *minimal surface.*

The parametric curves on a surface can be shown to form *an orthogonal net* in case $F=0$. Similarly, the parametric curves can be shown to form *a conjugate net* in case $D'=0$. Equation (14) shows that the parametric curves are the lines of curvature (supposed determinate) in case $F=D'=0$. It is known that the lines of curvature on a surface are indeterminate if the surface is a sphere or a plane. Thus one obtains the following characterization of the lines of curvature:

On an unspecialized surface in ordinary space the lines of curvature are the only orthogonal conjugate net.

If the lines of curvature are the parametric curves on a surface, then $F=D'=0$ and consequently the second of equations (11) becomes

$$x_{uv}=ax_u+bx_v\,, \tag{16}$$

where the coefficients a, b are given by the simple formulas

$$2a=(\log E)_v\,, \qquad 2b=(\log G)_u\,.$$

In this case it is easy to verify that the function ω defined by

$$2\omega = x^2 + y^2 + z^2 \tag{17}$$

is also a solution of equation (16). Conversely, since the homogeneous cartesian coordinates $x, y, z, 1$ are a special kind of projective homogeneous point coordinates, it follows that if these four coordinates satisfy an equation of Laplace

$$x_{uv} = cx + ax_u + bx_v \ , \tag{18}$$

then the parametric curves form a conjugate net, and $c=0$. Indeed, it is easy to verify by direct calculation that $D'=0$. Moreover, if the function ω also satisfies the same equation (18), then $F=0$ and the parametric curves form an orthogonal net. Thus the following theorem is demonstrated:

A necessary and sufficient condition that the parametric curves on a surface in ordinary space be the lines of curvature is that $x, y, z, 1, \omega$ satisfy an equation of Laplace of the general form (18)—*or what is equivalent, that x, y, z, ω satisfy an equation of Laplace of the particular form* (16).

45. Spheres and pentaspherical coordinates. After a review of the elements of the geometry of the sphere in the early part of this section, pentaspherical point coordinates will be defined, and some of the simpler facts of sphere geometry will be expressed in terms of these coordinates, which were first used by Darboux.

Let us recall the definition of a sphere and the criteria for the different types of spheres. In ordinary metric space the locus of a point whose cartesian coordinates x, y, z satisfy an equation of the form

$$a(x^2+y^2+z^2) + bx + cy + dz + e = 0 \ , \tag{19}$$

in which the coefficients are constants not all zero, is called *a sphere.* If $a \neq 0$, the sphere is a *proper sphere;* the coordinates of its center and the length R of its radius are given by the formulas

$$(-b/2a, \ -c/2a, \ -d/2a), \qquad R^2 = (b^2+c^2+d^2-4ae)/4a^2 \ .$$

If $a \neq 0$ and $R=0$, the sphere is called a *null sphere,* or *point-sphere;* such a sphere may be regarded as a single point. If $a=0$, the sphere is *composite,* one component being the plane

$$bx + cy + dz + e = 0 \ ,$$

and the other being the plane at infinity. If $R\neq 0$, and if a approaches zero, then R approaches infinity. Hence a plane is sometimes spoken of as a sphere with an infinite radius, the plane at infinity not being then considered as a component of the sphere.

The power of a point P with respect to a proper sphere is defined to be the product of the distances from P to any two points on the sphere collinear with P. Hence the power of the point P is equal, by elementary geometry, to the square of the length of a tangent from P to the sphere. The power of a point (x, y, z) with respect to the sphere (19) may be shown to be expressed by the formula

$$x^2+y^2+z^2+(bx+cy+dz+e)/a \; .$$

Two points are said to be *inverse* to each other with respect to a proper sphere in case* they are collinear with the center of the sphere and the product of their distances from the center is equal to the square of the radius of the sphere. The polar plane of each of two inverse points can be shown to pass through the other; therefore *two inverse points are conjugate* with respect to the sphere. In fact, *any two conjugate points collinear with the center of a sphere are inverse to each other with respect to the sphere.*

Pentaspherical point coordinates are defined as follows. Let us consider five spheres with equations of the form (19), the determinant of whose coefficients is not zero; and let us define the ratios of five numbers $x_1, \ldots, x_5$ by placing

$$\rho x_j = a_j(x^2+y^2+z^2)+b_jx+c_jy+d_jz+e_j \qquad (j=1, \ldots, 5) \; , \tag{20}$$

where ρ is a proportionality factor not zero, and x, y, z are the cartesian coordinates of a point P. Then $x_1, \ldots, x_5$ are by definition *pentaspherical coordinates*† of the point P. The reader should observe how the pentaspherical coordinates are connected with the powers of P with respect to the five spheres. It is clear that a different choice of the five spheres would result in a different pentaspherical coordinate system.

It is possible to attain a desirable degree of analytic simplicity by choosing *five mutually orthogonal spheres* as the fundamental spheres. We shall use hereinafter the special pentaspherical coordinates $x_1, \ldots, x_5$ whose ratios are defined by placing

$$\begin{cases} \rho x_1 = x \; , \qquad \rho x_2 = y \; , \qquad \rho x_3 = z \; , \\ \rho x_4 = i(x^2+y^2+z^2+1)/2, \qquad \rho x_5 = (x^2+y^2+z^2-1)/2 \; , \end{cases} \tag{21}$$

* Coolidge, 1916. 2, p. 227.

† Darboux, 1887. 1, p. 213.

where $i^2=-1$ and ρ is a proportionality factor not zero. It will be observed that the fundamental spheres for this system of pentaspherical coordinates are the three coordinate planes of the cartesian coordinate system regarded as spheres of infinite radii, the unit sphere $x^2+y^2+z^2-1=0$, and the imaginary unit sphere $x^2+y^2+z^2+1=0$, which are all mutually orthogonal. Moreover, direct calculation shows that the sum of the squares of these coordinates $x_1, \ldots, x_5$ is zero, so that *the special pentaspherical coordinates x of a point satisfy the quadratic equation*

$$\Sigma x^2=0 , \tag{22}$$

the summation here, as elsewhere in this section, ranging over the integers $1, \ldots, 5$.

Coordinates of a sphere may be defined as follows. A linear homogeneous equation

$$a_1x_1+a_2x_2+a_3x_3+a_4x_4+a_5x_5=0 ,$$

conveniently abbreviated into $\Sigma ax=0$, in pentaspherical coordinates x and with constant coefficients a not all zero, represents a sphere whose center in cartesian coordinates, and whose radius R, can be shown by direct methods to be given by the formulas

$$(-a_1/p,\ -a_2/p,\ -a_3/p) , \qquad R^2=\Sigma a^2/p^2 \qquad (p=a_5+ia_4) . \tag{23}$$

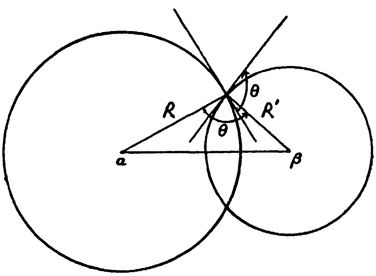

FIG. 34

The five coefficients a are called *coordinates of the sphere,* and one speaks of this sphere as *the sphere a.* The sphere is a point-sphere in case $\Sigma a^2=0$, $p\neq 0$; and is a plane in case $p=0$.

The angle θ between two spheres a, b with centers $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$ and radii R, R' can be calculated by means of the equation from the elementary geometry of a triangle, as in Figure 34,

$$\Sigma(\alpha-\beta)^2=R^2+R'^2-2RR'\cos\theta .$$

By the aid of the formulas (23) this equation can be reduced to

$$\Sigma ab=(\Sigma a^2\Sigma b^2)^{1/2}\cos\theta . \tag{24}$$

Therefore *a necessary and sufficient condition that two spheres a, b be orthogonal is*

$$\Sigma ab = 0 \ . \tag{25}$$

The condition for tangency of two spheres a, b is

$$(\Sigma ab)^2 = \Sigma a^2 \Sigma b^2 \ . \tag{26}$$

It is now possible to deduce* from the concluding theorem of the last section a very significant theorem. Let x, y, z be the non-homogeneous cartesian coordinates of a point P describing an unspecialized surface referred to its lines of curvature in ordinary metric space. Since $x, y, z, 1, \omega$ satisfy an equation of Laplace, it follows that the pentaspherical coordinates $x_1, \ldots, x_5$ of the point P, whatever be their proportionality factor ρ, satisfy an equation of Laplace of the form (18). The converse is also true. So we may state a theorem as follows:

A necessary and sufficient condition that a point, whose pentaspherical coordinates are functions of two variables u, v, may generate a surface referred to its lines of curvature is that the pentaspherical coordinates satisfy an equation of Laplace.

46. The correspondence between points of M_3 and points on a hyperquadric in S_4. The correspondence with which this section is concerned is defined as follows. *Let the pentaspherical coordinates x of a general point in ordinary metric space M_3 be interpreted as the projective homogeneous coordinates of a point in a linear space S_4; then to the points of space M_3 correspond the points on the hyperquadric Q_3 represented by equation* (22) *in the space S_4, corresponding points having the same or proportional coordinates.* Some of the essential features of this correspondence will be set forth in the present section, so that the correspondence may be applied in the next section to the theory of the lines of curvature. An account in English of this transformation may be found in the treatise† by Coolidge on the circle and the sphere.

To *a sphere* with the equation $\Sigma ax = 0$ in space M_3 corresponds the intersection of the hyperquadric Q_3 by the hyperplane represented in space S_4 by the same equation. The point P_a which is the pole of this hyperplane with respect to Q_3 is called *the second image* of the sphere. The sphere is a point-sphere in case the point P_a is on the hyperquadric Q_3. Consequently we have the theorem:

* *Ibid.*, p. 221; or 2d ed. (1914), p. 273.

† Coolidge, 1916. 2, p. 474.

To the spheres in ordinary metric space M_3 correspond as second images the points in space S_4, point-spheres in M_3 corresponding to points on the hyperquadric Q_3 in S_4.

Two spheres with the equations $\Sigma ax=0$, $\Sigma bx=0$ in space M_3 determine *a pencil of spheres,* the equation of a general sphere of the pencil being of the form

$$\Sigma(\lambda a+\mu b)x=0 \qquad (\lambda,\ \mu \text{ scalars})\ .$$

This pencil of spheres corresponds to the intersection of the hyperquadric Q_3 by the plane represented in space S_4 by the first two equations, and has for second image the polar line ab of the plane with respect to Q_3. Hence we state the theorem:

To the pencils of spheres in space M_3 correspond as second images the straight lines in space S_4.

A pencil of spheres in space M_3 belongs to one or the other of three classes according as the corresponding line ab intersects the hyperquadric Q_3 in one or the other of three possible ways. If the line ab intersects Q_3 in two distinct points, the corresponding pencil in space M_3 contains two point-spheres. The spheres of such a pencil cut their line of centers in pairs of points in an involution, of which the two double points are the two point-spheres. The point-spheres separate every pair of corresponding points of the involution harmonically, and consequently *the point-spheres are not only conjugate points but are actually inverse points with respect to every sphere of the pencil.* If, secondly, the line ab is tangent to the hyperquadric Q_3, so that the condition (26) is satisfied, there is only one point-sphere in the pencil, all of the spheres of the pencil being tangent to each other at this point. Finally, if the line ab is a generator of the hyperquadric Q_3, every sphere of the corresponding pencil is a point-sphere (see Ex. 1).

It follows from the preceding paragraph that when a line in space S_4 intersects the hyperquadric Q_3 in two points P_a, P_b, as indicated in Figure 35, any point P_c on the line ab, defined by

$$c=\lambda a+\mu b \qquad (\lambda,\ \mu \text{ scalars})\ , \tag{27}$$

is the second image of a sphere in the pencil containing the two point-spheres corresponding to the points P_a, P_b. These point-spheres are inverse points with respect to the sphere c. Conversely, it can be shown that *if two points a, b in space M_3 are inverse to each other with respect to a sphere c, these two points correspond to two points on the hyperquadric Q_3 which are collinear with the point P_c which is the second image of the sphere;* consequently a relation of the form (27) is satisfied. In fact, the two points a, b in space

M_3 can be regarded as point-spheres determining a pencil of spheres to which the given sphere belongs.

A circle in space M_3 can be regarded as the intersection of two spheres $\Sigma ax=0$, $\Sigma bx=0$. Therefore to a circle corresponds the conic of intersection of the hyperquadric Q_3 by a plane. The polar line ab of this plane with respect to Q_3 can be regarded as the second image either of the circle or of the pencil of spheres intersecting in the circle.

Pairs of circles in space M_3 can be classified according to the relations of two straight lines to each other and to the hyperquadric Q_3 in space S_4. We

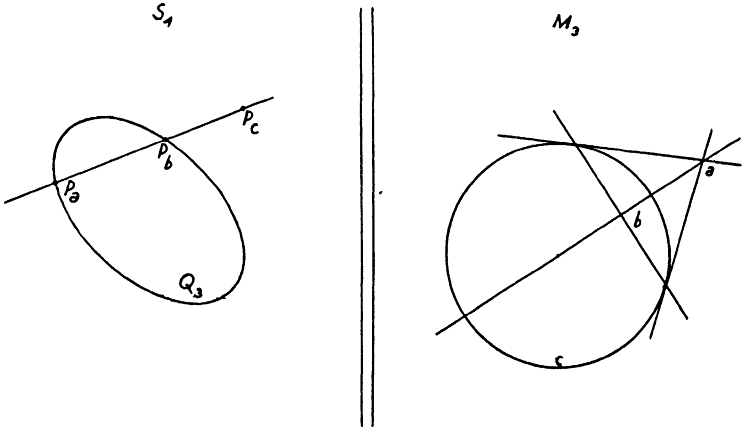

FIG. 35

shall not make an exhaustive classification, but shall confine ourselves to a few remarks. *If two circles in space M_3 correspond to two intersecting lines in space S_4, the point of intersection of these lines is the second image of a sphere containing both circles.* The plane of the two lines intersects the hyperquadric Q_3 in a conic. The nature of this conic, and the relations of the two lines to it, give information about two circles on the same sphere. If two circles correspond to two lines that do not intersect, the lines determine a hyperplane S_3 which cuts the hyperquadric Q_3 in a quadric surface. The nature of this quadric surface, and the relations of the two lines to it, give information about two circles not on the same sphere. Equation (25) shows that *two spheres in space M_3 are orthogonal in case their second images are conjugate points with respect to the hyperquadric Q_3.* Therefore the pole of the hyperplane S_3 with respect to Q_3 is the image of a sphere orthogonal to every sphere through either of the circles, and hence orthogonal to both circles. So we have the theorem:

In ordinary metric space there is just one sphere orthogonal to two circles that are not on a sphere.

When a curve in ordinary metric space M_3 is defined by its parametric equations in pentaspherical coordinates, a question arises as to the geometrical significance of the derivatives of the coordinates of a point on the curve. This geometrical significance is made clear by the following remarks. Let us consider in space M_3 a point P_x on a curve C whose parametric vector equation in pentaspherical coordinates is $x=x(t)$. *The derivatives x' of the*

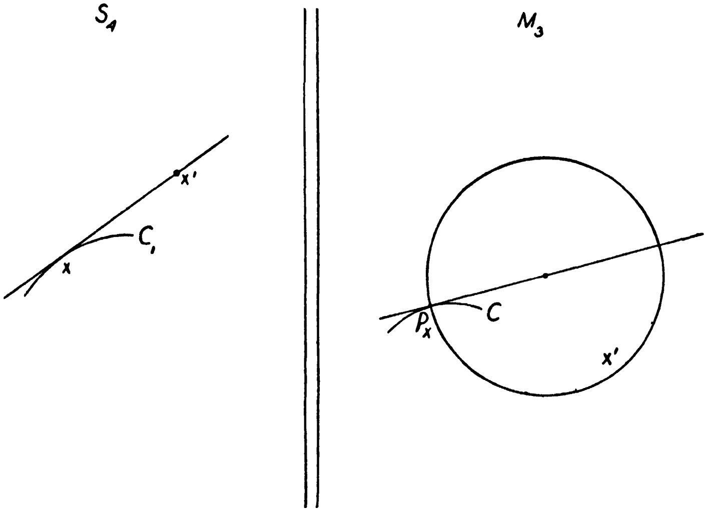

FIG. 36

coordinates x, calculated for the same value of t as the coordinates of the point P_x, are the coordinates of a sphere which has its center on the tangent at the point P_x of the curve C and which passes through P_x orthogonal to C. In order to prove these statements regarding the sphere x', let us refer to Figure 36 and consider the curve C_1 on the hyperquadric Q_3 in space S_4 that corresponds to the curve C, and on C_1 the point x that corresponds to P_x on C. At the point x the tangent of the curve C_1 is determined by the point x and the point x'. The point x', being in space S_4 and collinear with two consecutive points of the hyperquadric Q_3, is the second image of a sphere in space M_3 with respect to which two consecutive points of the curve C are inverse to each other. These two consecutive points determine the tangent of C at P_x, which therefore passes through the center of the sphere. Conse-

quently the sphere passes through the point of contact P_x of the tangent and is orthogonal to it, and hence is orthogonal to the curve C at the point P_x. This completes the proof.

The equation $\Sigma xx'=0$ derived from equation (22) shows analytically a fact just demonstrated above, namely, that the sphere x' passes through the point P_x. This sphere is a point-sphere in case $\Sigma x'^2=0$. Then the tangent of the curve C_1 lies entirely on the hyperquadric Q_3. In this case the tangent of the curve C is an isotropic line, and C is a minimal curve.

The next problem is *to determine analytically the osculating circle and the osculating sphere* at a point of a curve. Let us consider again a curve C whose parametric vector equation in pentaspherical coordinates is $x=x(t)$, and consider a point P_x on C. Any sphere $\Sigma ax=0$, whose coordinates a are functions of the parameter t, and which passes through the point P_x, contains two consecutive points of C at P_x if, and only if, $\Sigma ax'=0$. Such a sphere is tangent to C at P_x, and is orthogonal to the derivative sphere at this point. A two-point sphere a contains three consecutive points of C at P_x in case also $\Sigma ax''=0$. Such a sphere contains *the osculating circle* of C at P_x. There is a pencil of such spheres, every one of which has coordinates a satisfying the three conditions just written. Finally, a three-point sphere a contains four consecutive points of C at P_x in case also $\Sigma ax'''=0$. The sphere a is then *the osculating sphere* at the point P_x of the curve C, and its coordinates are found by solving the equations

$$\Sigma ax=0\ , \qquad \Sigma ax'=0\ , \qquad \Sigma ax''=0\ , \qquad \Sigma ax'''=0 \tag{28}$$

for the ratios of the coordinates a. The result for a general one of these coordinates can be written by means of a determinant of the fourth order,

$$a=(x,\, x',\, x'',\, x''')\ . \tag{29}$$

47. The lines of curvature. The correspondence studied in the last section will now be used to transform the theory of the lines of curvature on a surface S in ordinary metric space M_3 into the theory of a quadratic net N_x in space S_4. There are two spheres, called *principal spheres*, associated with each point P_x of the surface S. These spheres correspond to points in space S_4 that generate two nets in the relation of a transformation of Laplace. The relation of the sequence of Laplace thus determined to the sequence of Laplace determined by the quadratic net N_x that corresponds to the lines of curvature on the surface S will be examined. Then the interpretation in space M_3 of a transformation of Ribaucour applied to the quadratic net N_x will be explained.

Let us consider in ordinary metric space M_3 a surface S which is not a sphere, plane, or isotropic developable, and whose parametric vector equation in pentaspherical coordinates is $x=x(u, v)$, the lines of curvature being parametric. Since the coordinates x satisfy a quadratic equation of the form (22), and also satisfy an equation (18) of Laplace according to the last theorem of Section 45, it follows that *the lines of curvature on the surface S in space M_3 correspond to a quadratic net N_x on the hyperquadric Q_3 in space S_4.*

We shall now show that *at each point P_x of the surface S there exist two spheres each of which has the characteristic properties that it is tangent to S at P_x, contains the osculating circle of a line of curvature at P_x, and, as P_x varies along that line of curvature, has a characteristic circle which is tangent to S at P_x.* These spheres will be called *the principal spheres* of S at P_x. To make the demonstration* let us consider any curve C through the point P_x on the surface S. There is a one-parameter family of spheres tangent to S at P_x, and among them there is just one sphere that contains the osculating circle of C at P_x. The coordinates z of this sphere satisfy the conditions

$$(30) \qquad \Sigma zx=0\ , \qquad \Sigma zx_u=0\ , \qquad \Sigma zx_v=0\ , \qquad \Sigma zx''=0\ .$$

As the point P_x varies along the curve C, the characteristic circle of the sphere z is the intersection of this sphere and the sphere z'. Conditions necessary and sufficient that the sphere z', and hence the characteristic circle, be tangent to the surface S at the point P_x are

$$(31) \qquad \Sigma z'x=0\ , \qquad \Sigma z'x_u=0\ , \qquad \Sigma z'x_v=0\ .$$

The first of these conditions can be obtained by differentiating the first, and using the second and third, of equations (30). In the presence of the equations obtained by differentiating the second and third of (30), the second and third of the conditions (31) are equivalent to $\Sigma z(x_u)'=0$, $\Sigma z(x_v)'=0$. The last of equations (30) is a consequence of these two new conditions and the second and third of (30). Thus the following independent conditions have been imposed on the sphere z and the curve C:

$$(32) \quad \Sigma zx=0\ , \quad \Sigma zx_u=0\ , \quad \Sigma zx_v=0\ , \quad \Sigma z(x_u)'=0\ , \quad \Sigma z(x_v)'=0\ .$$

Elimination of z therefrom gives a condition on the curve C,

$$(33) \qquad (x,\ x_u,\ x_v,\ x_{uu}du+x_{uv}dv,\ x_{uv}du+x_{vv}dv)=0\ .$$

* Tzitzéica, 1924. 3, pp. 217–18.

No use has yet been made of the fact that the lines of curvature are parametric. By means of the Laplace equation (18), equation (33) can be reduced to

$$(34)\qquad (x,\ x_u,\ x_v,\ x_{uu},\ x_{vv})dudv=0\ .$$

Since the surface S is not a sphere or a plane, the coordinates x do not satisfy a linear algebraic equation with constant coefficients. Moreover, since S is not an isotropic developable, the coordinates x do not satisfy a second-order linear differential equation (see Ex. 4) that is independent of the Laplace equation (18). Therefore the first factor of equation (34) does not vanish, and the curve C must be one of the two parametric lines of curvature through the point P_x. Placing $v=$const. in equations (32) we may easily show that the resulting equations determine a unique sphere z. Similarly, placing $u=$const. in (32) we may show that the equations obtained determine a unique sphere z_{-1}. In fact, the two systems of equations in question are the following:

$$(35)\qquad \begin{cases} \Sigma zx=0\ , & \Sigma zx_u=0\ , & \Sigma zx_v=0\ , & \Sigma zx_{uu}=0\ ; \\ \Sigma z_{-1}x=0\ , & \Sigma z_{-1}x_v=0\ , & \Sigma z_{-1}x_u=0\ , & \Sigma z_{-1}x_{vv}=0\ . \end{cases}$$

Actual solution of the first system for the ratios of the coordinates z, and of the second system for z_{-1}, completes the existence proof and yields the following theorem.

The coordinates z of the principal sphere corresponding to the u-curve, and the coordinates z_{-1} of the principal sphere corresponding to the v-curve at a point P_x of a surface S, referred to its lines of curvature, are given by the formulas

$$(36)\qquad z=(x,\ x_u,\ x_v,\ x_{uu})\ ,\qquad z_{-1}=(x,\ x_v,\ x_u,\ x_{vv})\ .$$

Making use of the familiar notation for Laplace transforms, we may write

$$z=(x_{-2},\ x_{-1},\ x,\ x_1)\ ,\qquad z_{-1}=(x_{-1},\ x,\ x_1,\ x_2)\ .$$

An application of the theory of polar sequences of Laplace developed in Section 33 leads to the following conclusions (see Fig. 37). *As u, v vary, the point z, which is the second image in space S_4 of the principal sphere z, generates a net N_z which is one net of the polar sequence, with respect to the hyperquadric Q_3, of the sequence determined by the net N_x. The point z_{-1} generates the minus-first Laplace transform of the net N_z, as the notation indicates.*

We shall now prove that *the congruence of lines zz_{-1} is conjugate to the net N_x*. For the purpose of the demonstration* let us suppose that the pro-

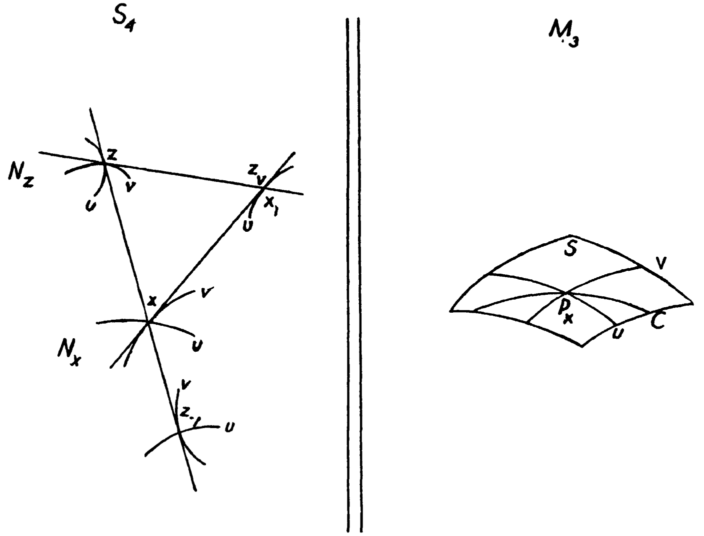

FIG. 37

portionality factor of z has been chosen so that $\Sigma z^2 = 1$. Differentiating this equation and the first three of (35) with respect to u and simplifying, we find

$$\Sigma z_u x = 0 \,, \qquad \Sigma z_u x_u = 0 \,, \qquad \Sigma z_u x_v = 0 \,, \qquad \Sigma z_u z = 0 \,,$$

so that $z_u = (x, x_u, x_v, z)$; we also have

$$\Sigma x x = 0 \,, \qquad \Sigma x x_u = 0 \,, \qquad \Sigma x x_v = 0 \,, \qquad \Sigma x z = 0 \,,$$

so that $x = (x, x_u, x_v, z)$. Consequently we obtain

$$z_u = \psi x \,, \tag{37}$$

where ψ is a proportionality factor. From the equations

$$\Sigma z_v x = 0 \,, \qquad \Sigma z_v x_u = 0 \,, \qquad \Sigma z_v x_{uu} = 0 \,, \qquad \Sigma z_v z = 0 \,,$$

* Tzitzéica, 1924. 3, pp. 218–19.

and from the two groups of equations

$$\Sigma x_v x=0\ , \qquad \Sigma x_v x_u=0\ , \qquad \Sigma x_v x_{uu}+a\Sigma x_u^2=0\ , \qquad \Sigma x_v z=0\ ;$$
$$\Sigma x x=0\ , \qquad \Sigma x x_u=0\ , \qquad \Sigma x x_{uu}+\ \Sigma x_u^2=0\ , \qquad \Sigma x z=0\ ;$$

which imply

$$\Sigma x_1 x=0\ , \qquad \Sigma x_1 x_u=0\ , \qquad \Sigma x_1 x_{uu}=0\ , \qquad \Sigma x_1 z=0\ ,$$

it follows that

$$z_v=\varphi x_1\ , \tag{38}$$

where φ is a proportionality factor. From equations (37), (38) it follows that the function φ is a solution of the adjoint of the Laplace equation (18), and the proof is finished by applying the theory developed at the close of Section 32 for determining all congruences conjugate to a given net.

Let us consider the net N_x on the hyperquadric Q_3 in space S_4 that corresponds to the (parametric) lines of curvature on a surface S_x in space M_3. Let us apply to the net N_x a transformation of Ribaucour to obtain a net N_y, also on the hyperquadric Q_3 of course, and corresponding to the lines of curvature on a surface S_y in space M_3. It is clear that *the lines of curvature on the surfaces S_x and S_y correspond*, and we are going to prove* that S_x, S_y *are the two sheets of the envelope of a two-parameter family of spheres, a general sphere of the family touching the two surfaces in corresponding points P_x, P_y.* For this purpose, referring to Figure 38, let us observe that, since the nets N_x, N_y are conjugate to one congruence, these nets are harmonic to another. Therefore, if the point of intersection of the lines xx_u, yy_u is denoted by ρ and that of the lines xx_v, yy_v by σ, the points ρ, σ are the focal points of a generator of the harmonic congruence. Using the notation of equation (V, 55), we find

$$\begin{cases} \rho=Xx_u-X_ux=-(Xy_u-X_uy)/Y\ , \\ \sigma=Xx_v-X_vx=-(Xy_v-X_vy)/Y\ . \end{cases} \tag{39}$$

The points ρ, σ are the second images of two spheres in space M_3, both of which are orthogonal to S_x at P_x and to S_y at P_y. For, the easily verified conditions

$$\Sigma\rho x=0\ , \qquad \Sigma\rho y=0\ , \qquad \Sigma\sigma x=0\ , \qquad \Sigma\sigma y=0 \tag{40}$$

* *Ibid.*, p. 220.

show that both spheres pass through P_x and through P_y; moreover, the sphere ρ is orthogonal to the u-curve at the point P_x, and is also orthogonal

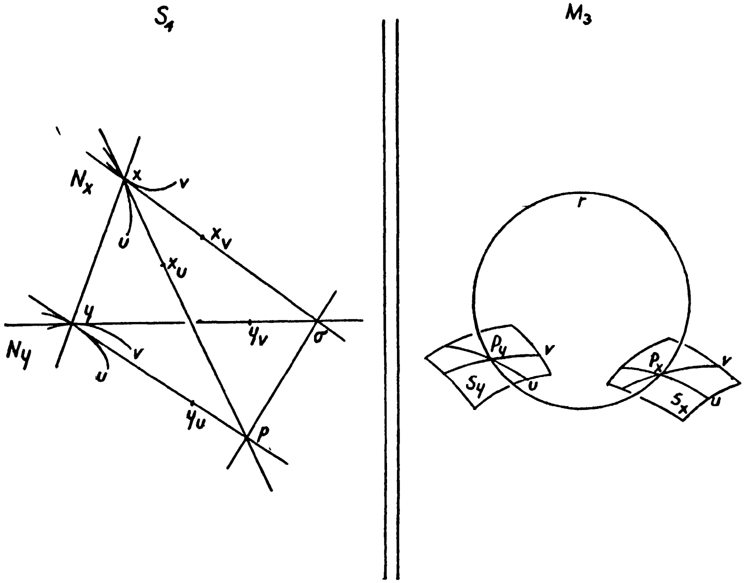

FIG. 38

to the u-curve at the point P_y, while the sphere σ is orthogonal to the v-curves at these points. The spheres ρ, σ intersect in a circle orthogonal to the surface S_x at P_x and to the surface S_y at P_y. The sphere r defined by

$$\Sigma rx = 0\,, \qquad \Sigma ry = 0\,, \qquad \Sigma r\rho = 0\,, \qquad \Sigma r\sigma = 0 \tag{41}$$

passes through the point P_x orthogonal to this circle, and likewise through the point P_y orthogonal to the same circle. Therefore *the sphere r is tangent to the surface S_x at the point P_x and to the surface S_y at the point P_y, so that, as u, v vary, the sphere r envelops the surfaces S_x, S_y.* This completes the proof.

We conclude with the definition of isothermic surfaces, and a theorem concerning them. The parametric curves on a surface in space M_3 have been seen to be an orthogonal net in case $F=0$. The parametric curves on a surface are said to form *an isothermally orthogonal net* in case

$$F = 0\,, \qquad (\log E/G)_{uv} = 0\,. \tag{42}$$

If the lines of curvature on a surface form an isothermally orthogonal net, the surface is by definition *isothermic.* Then equation (16) has equal Laplace-Darboux invariants. Consequently we may state the following conclusion:

A surface in ordinary metric space M_3 is isothermic if, and only if, its lines of curvature correspond to a conjugate net with equal invariants on the hyperquadric Q_3 in space S_4.

48. A local trihedron at a point of a surface in M_3. For the purpose of investigating certain aspects of the metric differential geometry of a surface in ordinary metric space M_3 it is convenient to introduce a local trihedron of reference at a general point of the surface. In this section we use the trihedron whose edges are the tangents of the lines of curvature and the normal at the point of the surface. This trihedron is in some respects analogous to the local tetrahedron employed in Chapter III, and seems especially suited for studying metric properties of certain projectively defined configurations associated with a point of a surface. For example, metric theorems are obtained concerning the quadrics of Darboux, union curves, and the axis and ray congruences of a conjugate net.

Let us consider a non-developable surface S, which is not a sphere, and which is defined analytically as in Section 44; let us suppose that the lines of curvature on S are parametric, so that $F=D'=0$, $DD''\neq 0$. Some of the formulas of that section simplify considerably under this assumption. For instance, equations (15) give

$$R_1=E/D\,,\qquad R_2=G/D''\,. \tag{43}$$

The last two of equations (11) become

$$X_u=-x_u/R_1\,,\qquad X_v=-x_v/R_2\,, \tag{44}$$

while the second of equations (11) becomes equation (16), since the Christoffel symbols (10) are now expressed by the simpler formulas

$$\begin{aligned}
2\{11\,,1\}&=(\log E)_u\,, &\qquad 2\{22,\,2\}&=(\log G)_v\,,\\
2\{21,\,1\}&=(\log E)_v\,, &\qquad 2\{12,\,2\}&=(\log G)_u\,,\\
2\{22,\,1\}&=-G_u/E\,, &\qquad 2\{11,\,2\}&=-E_v/G\,.
\end{aligned}$$

The equations (13) of Codazzi can now be written in the form

$$\begin{cases}2(1/R_1)_v=(1/R_2-1/R_1)(\log E)_v\,,\\ 2(1/R_2)_u=(1/R_1-1/R_2)(\log G)_u\,.\end{cases} \tag{45}$$

As *a local trihedron of reference* (see Fig. 39) at a point (x, y, z) of the surface S referred to its lines of curvature let us take the origin at this point, the ξ-axis along the u-tangent, the η-axis along the v-tangent, and the ζ-axis

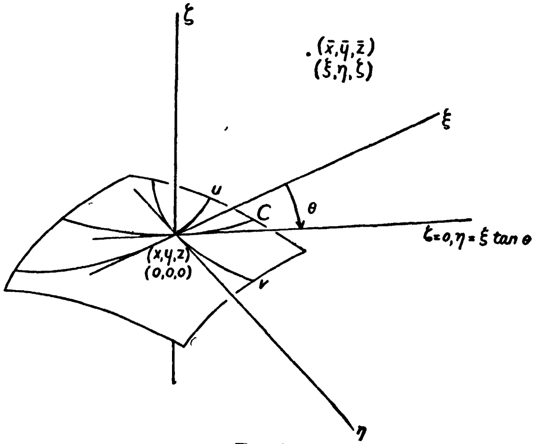

FIG. 39

along the surface normal through the point. If $\bar{x}$, $\bar{y}$, $\bar{z}$ are the general coordinates of a point having local coordinates ξ, η, ζ, the equations of transformation between the general and the local trihedrons can be written in the form

$$\left\{\begin{aligned} \bar{x}-x&=x_u\xi/E^{1/2}+x_v\eta/G^{1/2}+X\zeta\ , \\ \bar{y}-y&=y_u\xi/E^{1/2}+y_v\eta/G^{1/2}+Y\zeta\ , \\ \bar{z}-z&=z_u\xi/E^{1/2}+z_v\eta/G^{1/2}+Z\zeta\ . \end{aligned}\right. \tag{46}$$

The equations of the inverse transformation are

$$\begin{aligned} \xi&=(\bar{x}-x,\ x_v/G^{1/2},\ X)\ , \\ \eta&=(\bar{x}-x,\ X,\ x_u/E^{1/2})\ , \\ \zeta&=(\bar{x}-x,\ x_u/E^{1/2},\ x_v/G^{1/2})\ , \end{aligned}$$

in which parentheses indicate determinants of which only the first row is written in each case, the second and third rows being obtained from the first by replacing x, X by y, Y and then by z, Z.

The local equations of *the tangent line* at a point P of a curve C defined by $v=v(u)$ on a surface S are

$$\zeta=0\ , \qquad \eta=\xi\tan\theta\ , \tag{47}$$

where

$$\tan\theta = v'(G/E)^{1/2} \qquad (v' = dv/du) , \tag{48}$$

and θ is the angle that the tangent of the curve C makes with the ξ-axis. In particular, the equations of the asymptotic tangents at the point P are

$$\zeta = 0 , \qquad \xi^2/R_1 + \eta^2/R_2 = 0 . \tag{49}$$

The local equation of *the osculating plane* at the point P of the curve C is

$$(D + D''v'^2)(v'G^{1/2}\xi - E^{1/2}\eta) + s'^3\zeta/\rho = 0 \qquad (s' = ds/du) , \tag{50}$$

where ρ is defined by placing

$$\left\{\begin{aligned} s'^3/\rho = (EG)^{1/2}[v'' + \{11, 2\} + (2\{12, 2\} - \{11, 1\})v' \\ - (2\{21, 1\} - \{22, 2\})v'^2 - \{22, 1\}v'^3] . \end{aligned}\right. \tag{51}$$

Then $1/\rho$ is what is known as *the geodesic curvature* of the curve C at the point P. The geodesic curvature vanishes at every point of the curves called *geodesic curves.* In particular, for the geodesic curvatures $1/\rho_1$, $1/\rho_2$ of the parametric lines of curvature C_u, C_v at the point P we find* the expressions

$$1/\rho_1 = -(\log E)_v/2G^{1/2} , \qquad 1/\rho_2 = (\log G)_u/2E^{1/2} . \tag{52}$$

Equation (50) can be written also in the form

$$E^{1/2}(D + D''v'^2)(\xi \tan\theta - \eta) + s'^3\zeta/\rho = 0 . \tag{53}$$

For the purpose of obtaining metric results concerning some of the projectively defined configurations discussed in Chapter III, we shall employ *power series expansions for the local coordinates* ξ, η, ζ of a point near a point $P(0, 0, 0)$ on a surface S, in terms of the increments Δu, Δv corresponding to displacement on S from the point $(0, 0, 0)$ to the point (ξ, η, ζ). The calculation of these expansions is so similar to the calculation of the series (III, 15) that it will not be reproduced here. The results are as follows:

$$\left\{\begin{aligned} \xi &= E^{1/2}[\Delta u + (\{11, 1\}\Delta u^2 + 2\{12, 1\}\Delta u\Delta v + \{22, 1\}\Delta v^2)/2 + \cdots] , \\ \eta &= G^{1/2}[\Delta v + (\{11, 2\}\Delta u^2 + 2\{12, 2\}\Delta u\Delta v + \{22, 2\}\Delta v^2)/2 + \cdots] , \\ \zeta &= (D\Delta u^2 + D''\Delta v^2)/2 + [(D_u + \{11, 1\}D)\Delta u^3 + 3\{12, 1\}D\Delta u^2\Delta v \\ &\qquad + 3\{12, 2\}D''\Delta u\Delta v^2 + (D''_v + \{22, 2\}D'')\Delta v^3]/6 + \cdots . \end{aligned}\right. \tag{54}$$

* Eisenhart, 1909. 1, p. 134.

The equation of *any quadric having contact of the second order* with the surface S at the point P is easily shown to be of the form

$$\xi^2/R_1+\eta^2/R_2+2\zeta(-1+k_2\xi+k_3\eta+k_4\zeta)=0\ , \tag{55}$$

where k_2, k_3, k_4 are arbitrary. This quadric is not a parabola if the function f defined by

$$f=(2k_4-R_1k_2^2-R_2k_3^2)/R_1R_2 \tag{56}$$

is not zero; then the center of the quadric is the point

$$(-k_2/fR_2,\ -k_3/fR_1,\ 1/fR_1R_2)\ . \tag{57}$$

The center is on the normal of the surface at the point P in case $k_2=k_3=0$.

The equation of *the curves of Darboux* on a surface is found to be

$$D\mathfrak{C}'du^3-3D\mathfrak{B}'du^2dv-3D''\mathfrak{C}'dudv^2+D''\mathfrak{B}'dv^3=0\ , \tag{58}$$

where the functions $\mathfrak{B}'$, $\mathfrak{C}'$ are given by the formulas

$$8\mathfrak{B}'=(\log R_1^3/R_2)_v\ , \qquad 8\mathfrak{C}'=(\log R_2^3/R_1)_u\ . \tag{59}$$

The quadric (55) is a *quadric of Darboux* in case

$$4E^{1/2}k_2=(\log K)_u\ , \qquad 4G^{1/2}k_3=(\log K)_v\ , \tag{60}$$

where K is the total curvature defined by the first of equations (15). Hence we deduce the theorem*:

The centers of the quadrics of Darboux at each point P of a surface S are on the normal of S at P if, and only if, the surface S has constant total curvature.

Any line l_1 through a point $P(0, 0, 0)$ and not in the tangent plane, $\zeta=0$, can be regarded as joining P to a point $(-a,\ -b,\ 1)$, where a, b are functions of u, v. By means of equation (50) *the union curves* of the congruence Γ_1 generated by l_1 are found to have the differential equation

$$\left\{\begin{aligned} v''=&-Db/G^{1/2}-\{11,\ 2\}+(Da/E^{1/2}+\{11,\ 1\}-2\{12,\ 2\})v' \\ &+(-D''b/G^{1/2}-\{22,\ 2\}+2\{21,\ 1\})v'^2+(D''a/E^{1/2}+\{22,\ 1\})v'^3\ . \end{aligned}\right. \tag{61}$$

If the line l_1 is normal to the surface then $a=b=0$ and the union curves are the geodesics on the surface. Thus we have proved the theorem:

* Fubini and Čech, 1926. 1, pp. 177–78.

The union curves of the normal congruence of a surface are the geodesics on the surface.

The equations of the osculating planes of the lines of curvature C_u, C_v at a point P are respectively

$$\rho_1\eta - R_1\zeta = 0 , \qquad \rho_2\xi + R_2\zeta = 0, \tag{62}$$

where R_1, R_2 are given by equations (43), and ρ_1, ρ_2 by (52). Therefore *the axis of a point $P(0, 0, 0)$ on a surface, with respect to the lines of curvature,* joins P to the point $(-a, -b, 1)$ for which

$$a = R_2/\rho_2 , \qquad b = -R_1/\rho_1 . \tag{63}$$

Moreover, *the ray of the point P with respect to the lines of curvature* intersects the tangents of the lines of curvature C_u, C_v through P in the points whose general coordinates are given by the respective expressions

$$x - \rho_2 x_u/E^{1/2} , \qquad x + \rho_1 x_v/G^{1/2} , \tag{64}$$

as can be verified by showing that the v-derivative of the first expression is proportional to x_u, and that the u-derivative of the second is proportional to x_v. Additional results related to this section may be found in Exercises 9, . . . , 13, 16, 21, 22, 24.

49. The transformation between cartesian coordinates and Fubini's normal coordinates. Besides the method used in the last section for studying metric properties of projectively defined configurations in the geometry of surfaces in ordinary space, there is still another method, which will be employed in this section. This method rests on the following observation. If x, y, z are the ordinary cartesian coordinates of a point P on a surface, then the homogeneous cartesian coordinates x, y, z, 1 can be regarded as a special kind of projective homogeneous coordinates of P, and there exists a function λ such that λx, λy, λz, λ are Fubini's normal coordinates of P, which satisfy equations (III, 6) when the asymptotic curves on the surface are parametric. After simplifying the formulas of Section 44 by choosing the asymptotic curves as parametric, we shall actually find a formula for the factor λ. The result will be applied in a brief discussion of the projective normal and the first directrix of Wilczynski.

Let us consider a non-ruled surface S defined as in Section 44, and let us suppose that the asymptotic curves on S are parametric, so that $D = D'' = 0$, $D' \neq 0$. Some of the formulas of that section simplify considerably under this assumption. If a function k^2 is defined by placing

$$H^2 = k^2 D'^2 \qquad (H^2 = EG - F^2) , \tag{65}$$

then for the total curvature K we have the expressions

$$K=-D'^2/H^2=-1/k^2\ . \tag{66}$$

The equations (13) of Codazzi become

$$(\log D')_u=\{11,\ 1\}-\{12,\ 2\}\ ,\qquad (\log D')_v=\{22,\ 2\}-\{21,\ 1\}\ , \tag{67}$$

and it is easy, by means of the formulas (10), to verify the identities

$$(\log H)_u=\{11,\ 1\}+\{12,\ 2\}\ ,\qquad (\log H)_v=\{22,\ 2\}+\{21,\ 1\}\ . \tag{68}$$

Consequently we obtain

$$(\log k)_u=2\{12,\ 2\}\ ,\qquad (\log k)_v=2\{21,\ 1\}\ . \tag{69}$$

We proceed to discover the formula for the multiplier λ that converts cartesian coordinates into Fubini's normal coordinates. The effect of a transformation $x=\lambda\bar{x}$ on the coefficients of equations (III, 6) is given by equations (III, 4). Let the coordinate system be specialized by choosing for the tetrahedron of reference the tetrahedron whose faces are the three cartesian coordinate planes and the plane at infinity, and by selecting the unit point appropriately, so that the coordinate system is cartesian. Further, let the coordinates $\bar{x}$ be the cartesian coordinates $x,\ y,\ z,\ 1$. Since λ must now be a solution of equations (III, 6), it follows from the first two of equations (III, 4) that $\bar{p}=\bar{q}=0$. Moreover, the first and third of equations (11) and the last three of equations (III, 4) now give

$$\left\{\begin{array}{ll}\{11,\ 1\}=\bar{\theta}_u=\theta_u-2(\log\lambda)_u\ , & \{11,\ 2\}=\bar{\beta}=\beta\ ,\\ \{22,\ 1\}=\bar{\gamma}=\gamma\ , & \{22,\ 2\}=\bar{\theta}_v=\theta_v-2(\log\lambda)_v\ .\end{array}\right. \tag{70}$$

Equations (68), (69), (70) imply

$$(\log H)_u=\theta_u-2(\log\lambda)_u+(\log k)_u/2\ ,$$
$$(\log H)_v=\theta_v-2(\log\lambda)_v+(\log k)_v/2\ .$$

These equations determine the function λ, except for a constant factor; integrating them we obtain the desired formula, which can be written in two equivalent forms,

$$\lambda=e^{\theta/2}/D'^{1/2}k^{1/4}=(\{11,\ 2\}\{22,\ 1\})^{1/2}/(D'H)^{1/4}\ . \tag{71}$$

We have thus established the result:

Multiplication of the cartesian coordinates x, y, z, 1 *by the factor* λ *given in* (71) *converts them into Fubini's normal coordinates.*

As an application of this result, we shall show how it can be used to study *metric properties of the projective normal.* When the coordinates are Fubini's normal coordinates, the projective normal at a point x of a surface joins this point to the point x_{uv}. After a transformation $x=\lambda\bar{x}$, the projective normal joins the point $\lambda\bar{x}$ to the point $(\lambda\bar{x})_{uv}$. If λ is given by (71), and if the coordinates $\bar{x}$ are the cartesian coordinates $x, y, z, 1$, it follows that the projective normal joins the point $(x, y, z, 1)$ to the point $[(\lambda x)_{uv}, (\lambda y)_{uv}, (\lambda z)_{uv}, \lambda_{uv}]$. *The point at infinity on the projective normal* has the homogeneous cartesian coordinates

$$[x_{uv}+(\log \lambda)_v x_u+(\log \lambda)_u x_v\ ,\ y_{uv}+(\log \lambda)_v y_u+(\log \lambda)_u y_v\ ,$$
$$z_{uv}+(\log \lambda)_v z_u+(\log \lambda)_u z_v,\ 0]\ .$$

The direction cosines of the projective normal are proportional to the first three of these coordinates, and hence are proportional to the expression

$$(\log \lambda k^{1/2})_v x_u+(\log \lambda k^{1/2})_u x_v+D'X$$

and the two other expressions obtained by substituting y, Y and z, Z in place of x, X. The cartesian equations of the projective normal could easily be written, since its direction cosines and the coordinates of a point on it are known.

A condition necessary and sufficient that the projective normal may coincide with the metric normal is found, by equating to zero the coefficients of x_u, x_v in the last expression above, to be

$$\lambda k^{1/2}=c \qquad (c=\text{const.})\ . \tag{72}$$

Comparing the value of λ thus determined with the value given by (71) we arrive at a condition on a surface at every point of which the two normals coincide, namely,

$$c^2D'=e^{\theta}k^{1/2}\ .$$

More generally, the direction cosines of any canonical line of the first kind are proportional to the expression

$$[k'\psi+(\log \lambda k^{1/2})_v]x_u+[k'\varphi+(\log \lambda k^{1/2})_u]x_v+D'X$$

and the two symmetrical expressions in y, Y and z, Z, in which k' is the constant k of Section 20 and φ, ψ are defined by equations (III, 9). This

line coincides with the metric normal in case the coefficients of x_u and x_v vanish. If the canonical line* is the directrix of Wilczynski, for which $k' = -1/2$, the conditions that this line may coincide with the metric normal can be written in the form

$$(\log k)_v = 2(\log \beta D')_v \,, \qquad (\log k)_u = 2(\log \gamma D')_u \,. \tag{73}$$

It follows that $(\log \beta/\gamma)_{uv} = 0$. A surface for which this partial differential equation is satisfied is called *isothermally asymptotic*. After a transformation of parameters we have

$$\beta = \gamma \,, \qquad k = c\beta^2 D'^2 \qquad (c = \text{const.}) \,.$$

50. Affine geometry of surfaces. In this section we do not attempt to establish a theory of the invariants of surfaces under the general affine transformation in ordinary space, but consider only such affine transformations as leave one point, the origin, invariant. A completely integrable system of differential equations is set up which defines a surface except for an affine transformation of this kind, and an interesting property of a pair of integral surfaces of this system of equations is established. A special class of surfaces, each of which has a property called *the property of Tzitzéica*, is defined, and the section closes with a formula connecting the coordinates used in the affine theory with Fubini's normal coordinates.

Let us first of all prepare to exlude a certain special class of surfaces. If the cartesian coordinates x, y, z of a point on a surface S in ordinary space are given as functions of two variables u, v and satisfy a differential equation of the form

$$ax_u + bx_v + cx = 0 \,,$$

whose coefficients a, b, c are scalar functions of u, v and are not all zero, then integration of this equation shows, as in Section 8, that the coordinates x, y, z can be expressed in the form

$$x = \varphi f_1(t) \,, \qquad y = \varphi f_2(t) \,, \qquad z = \varphi f_3(t) \,, \tag{74}$$

where φ, t are functions of u, v. Elimination of φ, t leads to a homogeneous algebraic equation in x, y, z. Therefore *the surface S is a cone with its vertex at the origin.* Such surfaces will be excluded from the following discussion, unless the contrary is explicitly stated.

* Fubini, 1927. 8.

When the coordinates x, y, z of a general point on an unspecialized surface S are given as functions of two variables u, v it is possible to determine* the coefficients of the equations

$$\left\{\begin{aligned} x_{uu} &= px+ax_u+\beta x_v\ , \\ x_{uv} &= cx+ax_u+bx_v\ , \\ x_{vv} &= qx+\gamma x_u+\delta x_v \end{aligned}\right. \tag{75}$$

so that each of x, y, z will satisfy each of the equations. For example, if each of x, y, z is substituted in turn in the first equation with the coefficients p, a, β regarded as unknown, the resulting system of three linear algebraic equations can be solved uniquely for these coefficients. The other coefficients can be determined similarly. Then S is an integral surface of the system (75). This system is of the same form as the system (IV, 26), of which the integrability conditions are equations (IV, 27).

A transformation whose equations in cartesian coordinates are linear and homogeneous with constant coefficients, of the form

$$\bar{x}=a_1x+b_1y+c_1z\ , \qquad \bar{y}=a_2x+b_2y+c_2z\ , \qquad \bar{z}=a_3x+b_3y+c_3z\ ,$$

is an affine transformation leaving the origin invariant. Any surface obtained from an integral surface of system (75) by such a transformation is also an integral surface of the system; conversely, system (75) with its integrability conditions satisfied determines a surface except for such an affine transformation.

Using the cosines X, Y, Z defined by the formulas (6) and the second fundamental coefficients D, D', D'' defined by (9) we find, for an integral surface S of system (75),

$$D=p\Sigma Xx\ , \qquad D'=c\Sigma Xx\ , \qquad D''=q\Sigma Xx\ . \tag{76}$$

If $\Sigma Xx=0$, the surface S is a plane passing through the origin. Such surfaces being excluded, the differential equation (8) of *the asymptotic curves* on the surface S becomes

$$pdu^2+2cdudv+qdv^2=0\ . \tag{77}$$

If the asymptotic curves on the surface S (supposed non-developable) are parametric, then $p=q=0$, $c\neq 0$, and the first two of the integrability conditions (IV, 27) become

$$(\log c)_u=a-b\ , \qquad (\log c)_v=\delta-a\ . \tag{78}$$

* Tzitzéica, 1924. 3, p. 244.

Consequently we have $(a-b)_v=(\delta-a)_u$; but the third and fourth of the conditions (IV, 27) imply $(a+b)_v=(\delta+a)_u$. One deduces immediately the equations

$$a_v=\delta_u\ , \qquad b_v=a_u\ , \tag{79}$$

and hence a theorem:

When the asymptotic curves on a non-developable integral surface of system (75) are parametric, the Laplace equation in this system has equal invariants.

We shall now establish a metric property* of any pair of non-developable integral surfaces S, S' of system (75). Let d, d' denote the distances from the origin to the tangent planes at corresponding points of S, S', and let K, K' be the total curvatures of these surfaces at these points. We propose to prove that *the surfaces S, S' have the property expressed by the equation*

$$K/d^4 \div K'/d'^4=\text{const.} \tag{80}$$

First of all, we observe that the distance d from the origin to the tangent plane (2) at a point (x, y, z) of the surface S is given by

$$\pm d=\Sigma Xx=D'/c=(x,\ x_u,\ x_v)/H\ . \tag{81}$$

If the surface S is referred to its asymptotic curves, the total curvature K of S at the point (x, y, z) is given by

$$K=-D'^2/H^2=-c^2d^2/H^2\ . \tag{82}$$

Consequently we find

$$K/d^4=-c^2/d^2H^2=-c^2/(x,\ x_u,\ x_v)^2\ . \tag{83}$$

Let us define a function n by placing

$$K/d^4=-1/n^4\ , \tag{84}$$

so that

$$1/n^2=\pm c/(x,\ x_u,\ x_v)\ . \tag{85}$$

Logarithmic differentiation of equation (85) and use of equations (75), (78) lead to the conditions

$$(\log n)_u=b\ , \qquad (\log n)_v=a\ . \tag{86}$$

* *Ibid.*, p. 246.

Consequently log n is determined by system (75), except for an additive constant, and the function n is determined except for a multiplicative constant. Equation (84) now shows that the ratio K/d^4 is determined by system (75), except for a constant factor. The ratio K'/d'^4 is determined by system (75) *in the same way*, except for a constant factor. Therefore the statement made in equation (80) is true, as was to be proved.

It follows from the preceding paragraph that if $K/d^4=\text{const.}$, then also $K'/d'^4=\text{const.}$ Therefore *the property expressed by the equation* $K/d^4=$ const. *is of an affine nature.* Equations (84), (86) show that *all non-developable integral surfaces of system* (75) *have this property if, and only if,* $a=b=0$. We shall now discover an affine geometric characterization of such surfaces. The parametric equations of the ruled surface R_u of u-tangents constructed at the points of a fixed v-curve of such a surface are

$$\bar{x}=x+tx_u\,,\qquad \bar{y}=y+ty_u\,,\qquad \bar{z}=z+tz_u\,,$$

wherein $u=\text{const.}$ and the parameters are t, v. The equation

$$(\xi-\bar{x},\ \bar{x}_t,\ \bar{x}_v)=0$$

of the tangent plane at a point $(\bar{x}, \bar{y}, \bar{z})$ of the surface R_u reduces to

$$(\xi-x,\ x_u,\ x_v)+t(\xi-x,\ x_u,\ x_{uv})=0\ .$$

Consequently the equation of the tangent plane at a point at infinity (or *an asymptotic tangent plane*) of the surface R_u is

$$(\xi-x,\ x_u,\ x_{uv})=0\ . \tag{87}$$

This plane passes through the origin if, and only if, $b=0$. Thus we know a geometrical meaning for the condition $b=0$. Similarly, using the ruled surface R_v, defined as R_u but with u and v interchanged, we obtain a meaning for the condition $a=0$. Therefore *a surface has the property expressed by the equation* $K/d^4=$*const. if, and only if, each ruled surface of the tangents of the asymptotic curves of one family, constructed at the points of a fixed asymptotic curve of the other family, cuts the plane at infinity in a curve such that the tangent planes of the ruled surface at the points of this curve pass through the origin.* But these tangent planes envelop *the asymptotic developable* of the ruled surface. Moreover, we shall call the property expressed by the equation $K/d^4=\text{const.}$ *the property* of Tzitzéica,* because it has been studied so ex-

* *Ibid.*, p. 250.

tensively by him. We are enabled thus to conclude with the following proposition.

A non-developable surface in ordinary space has the property of Tzitzéica if, and only if, the asymptotic developables of the ruled surfaces of tangents of the asymptotic curves of each family, circumscribing the surface along the asymptotic curves of the other family, are cones all of which have a common vertex.

The same method that was used in the preceding section can be used here to establish the transformation between the coordinates x, y, z of the present section and Fubini's normal coordinates. We take the asymptotic curves as parametric, and suppose that the surfaces considered are not ruled. If λ is the factor by which the homogeneous coordinates x, y, z, 1 of the affine theory must be multiplied in order to convert them into Fubini's normal coordinates, we find

$$\lambda^2 = he^{\theta}/cn \qquad (h = \text{const.};\ \theta = \log \beta\gamma) , \tag{88}$$

where n satisfies the conditions (86).

51. Surfaces with indeterminate directrix curves. The directrix curves on a surface S in ordinary projective space are defined to be the curves in which the developables of the first directrix congruence of S intersect S; the same curves also correspond to the developables of the second directrix congruence of the surface S. The purpose of this section is, first of all, to lay the foundations for a purely projective theory of those surfaces on each of which the directrix curves are indeterminate. On such a surface every curve is a directrix curve, and therefore corresponds to a developable of each directrix congruence. Then by means of the transformation introduced at the close of the last section, which permits to pass from the projective theory to the special affine theory of surfaces, it will be proved that every surface, not of a certain special type, whose directrix curves are indeterminate can be projected into a surface having the property of Tzitzéica.

The differential equation of the directrix curves on an integral surface S of system (III, 6) is equation (III, 39) when the functions a, b therein have the values shown in equations (III, 48). These curves are indeterminate in case the three coefficients in their equation vanish. From the vanishing of the second coefficient we obtain $(\log \beta/\gamma)_{uv} = 0$, so that by a change of parameters we have $\beta = \gamma$. This means that the surface S is *isothermally asymptotic*. The vanishing of the other two coefficients gives two conditions which, when $\beta = \gamma$, reduce to

$$p = f_{uu} - f_u^2/3 - \beta f_v, \qquad q = f_{vv} - f_v^2/3 - \beta f_u , \tag{89}$$

where for abbreviation we have defined f by placing

$$2f=3\log\beta\ . \tag{90}$$

Substitution of these expressions for the coefficients p, q in the first two of the integrability conditions (III, 10) gives two partial differential equations for the function f, each of which can be integrated once. The result of this integration is that β must satisfy the equation

$$(\log\beta)_{uv}=\beta^2+k/\beta \qquad (k=\text{const.})\ . \tag{91}$$

The third integrability condition is satisfied identically. Thus we are led to the following conclusion.

If β is a solution of equation (91), *if p, q are defined by equations* (89), *and if $\gamma=\beta$, then all of the integrability conditions* (III, 10) *are satisfied, and every integral surface of system* (III, 6) *is a surface* with indeterminate directrix curves.*

Equation (III, 40) shows that, when the directrix curves are indeterminate, the two foci of the directrix d_1 of the first kind at a point x coincide in the point z given by

$$z=(f_uf_v-3\beta^2/2-k/2\beta)x-f_vx_u-f_ux_v+x_{uv}\ . \tag{92}$$

That the point z is fixed is evident geometrically, and is demonstrated analytically by the formulas

$$3z_u=-f_uz\ , \qquad 3z_v=-f_vz\ , \tag{93}$$

which follow from differentiating z and reducing the results. *All the directrices of the first kind pass through the fixed point z.* Dually, *all the directrices of the second kind lie in a fixed plane.* This plane is determined by the three points

$$x_u-f_ux\ , \qquad x_v-f_vx\ , \qquad x_{uv}-f_{uv}x-f_ux_v\ .$$

Therefore the equation of this plane referred to the local tetrahedron x, x_u, x_v, x_{uv}, with suitably chosen unit point, is

$$x_1+f_ux_2+f_vx_3+(f_uf_v+3\beta^2/2+3k/2\beta)x_4=0\ . \tag{94}$$

The point y in which the line xz meets the fixed plane (94) is given by

$$y=z-kx/\beta\ . \tag{95}$$

* Wilczynski, 1914. 3.

It is evident now that *the fixed point and plane are in united position if, and only if,* $k=0$. In this case equation (91) shows that the asymptotic curves on the surface S belong to linear complexes. In what follows we shall suppose that $k\neq 0$.

We shall next show that *a surface S with indeterminate directrix curves has the property that each ruled surface of the tangents of the asymptotic curves of one family, constructed at the points of a fixed asymptotic curve of the other family, intersects the fixed plane* (94) *in a curve such that the tangent planes of the ruled surface at points of this curve envelop a cone with its vertex at the fixed point z.* For this purpose it is useful to observe that the u-tangent at a point x of the surface S meets the fixed plane (94) at the point x_u-f_ux. The tangent plane at this point of the ruled surface generated by the u-tangent as v varies is determined by this point, the point x, and the point $(x_u-f_ux)_v$, or else by the points x, x_u, $x_{uv}-f_ux_v$. To complete the proof it is sufficient to remark that the point z given by equation (92) is linearly dependent on these points.

The theory of surfaces with indeterminate directrix curves, as outlined in the foregoing paragraphs of this section, has been purely projective. But we are now going to employ the transformation mentioned at the end of the preceding section to pass to the affine geometry of these surfaces. In particular, we shall show that *every surface with indeterminate directrix curves* $(k\neq 0)$ *can be projected into a surface having the property of Tzitzéica,* by choosing the fixed point as the origin and projecting the fixed plane into the plane at infinity.

After a transformation $x=\lambda\bar{x}$, equation (92) becomes

$$(96)\quad \begin{cases} z/\lambda=[f_uf_v-3\beta^2/2-k/2\beta-f_u(\log\lambda)_v-f_v(\log\lambda)_u+\lambda_{uv}/\lambda]\bar{x} \\ \qquad\qquad -[f_v-(\log\lambda)_v]\bar{x}_u-[f_u-(\log\lambda)_u]\bar{x}_v+\bar{x}_{uv}\ . \end{cases}$$

The points x_u-f_ux, x_v-f_vx, which are on the directrix d_2 of the second kind at the point x, are represented by the expressions

$$(97)\qquad \bar{x}_u-[f_u-(\log\lambda)_u]\bar{x}\ ,\qquad \bar{x}_v-[f_v-(\log\lambda)_v]\bar{x}$$

after the transformation. If the function λ is defined by equation (88), if the coordinate system is specialized so that it is cartesian, if the coordinates $\bar{x}$ are homogeneous cartesian coordinates x, y, z, 1, and if the fixed point z of equation (96) is chosen as the origin, then the non-homogeneous cartesian coordinates x, y, z must satisfy the equation that results when the right member of equation (96) is set equal to zero. Finally, if the fixed plane containing the directrices of the second kind is projected into the plane at

infinity, the two expressions (97) must vanish when 1 is substituted therein in place of $\bar{x}$; consequently we have

$$(98) \qquad (\log \lambda)_u = f_u , \qquad (\log \lambda)_v = f_v .$$

The effect of the transformation under consideration on equations (III, 6) is easily calculated. Thus we find that the coordinates x, y, z satisfy a system of equations* of the form (75), namely,

$$(99) \qquad \begin{cases} x_{uu} = -(\log \beta)_u x_u + \beta x_v , \\ x_{uv} = -kx/\beta , \\ x_{vv} = \beta x_u - (\log \beta)_v x_v , \end{cases}$$

wherein β is a solution of equation (91) with $k \neq 0$. Since the terms in x_u, x_v are missing from the second of equations (99), it is seen that $a = b = 0$ in the notation of system (75). Therefore *all integral surfaces of system* (99) *have the property of Tzitzéica.* Thus we have completed the proof promised in the last paragraph.

EXERCISES

1. If every sphere of a pencil in ordinary metric space M_3 is a point-sphere, the line of centers is an isotropic (minimal straight) line. Therefore the isotropic lines of space M_3 correspond to the rectilinear generators of the hyperquadric Q_3 in space S_4.

2. The circle common to all the spheres of a pencil of the first or second class in space M_3 lies in a plane (called *the radical plane* of the pencil). If this plane is taken as the plane $x = 0$, the equation of a general sphere of the pencil is

$$x^2 + y^2 + z^2 - 2gx + c = 0 \qquad (c = \text{const.};\ g = \text{parameter}) .$$

The circle is real, a point-circle, or imaginary according as the pencil contains two imaginary, one real, or two real point-spheres. Prove analytically that in the last case the polar plane of each point-sphere with respect to any sphere of the pencil passes through the other point-sphere.

3. Two spheres are orthogonal in case one of them contains two points that are inverse points with respect to the other sphere.

4. If the pentaspherical coordinates x of a point on a surface S referred to its lines of curvature satisfy an equation of Laplace and an independent differential equation of the second order, the corresponding net N_x on the hyperquadric Q_3 lies on a developable surface, and S is an isotropic developable; that is, S consists of isotropic lines tangent to a minimal curve.

* Fubini and Čech, 1926. 1, p. 167.

5. The sphere r defined by equations (41) has for second image in space S_4 a point which, as u, v vary, generates a net which has a Laplace equation of the form $r_{uv}=Ar_u+Br_v$ when the proportionality factor of r is chosen so that $\Sigma r^2=1$.

TZITZÉICA, 1924. 3, p. 221

6. A transformation by inversion from a surface S_x to a surface S_y with respect to a sphere z in space M_3 corresponds to a transformation of Ribaucour from a net N_x to a net N_y on Q_3 in S_4, the conjugate congruence of this transformation being a bundle of lines with center at the point z. Hence lines of curvature are preserved under a transformation by inversion.

7. An inversion with respect to a sphere in space M_3 carries an isothermic surface into an isothermic surface.

8. When the lines of curvature are parametric on a surface S_x in space M_3, if the v-curves are spherical, the Laplace sequence determined by the corresponding net N_x in space S_4 terminates in the v-direction according to the case of Goursat.

TZITZÉICA, 1924. 3, p. 239

9. The differentials of the direction cosines of the three edges of the local trihedron composed of the tangents of the lines of curvature and the normal at a point of a surface are given by the formulas

$$d(x_u/E^{1/2})=[(\{11,\ 2\}du+\{12,\ 2\}dv)x_v+DX\ du]/E^{1/2}\ ,$$
$$d(x_v/G^{1/2})=[(\{21,\ 1\}du+\{22,\ 1\}dv)x_u+D''X\ dv]/G^{1/2}\ ,$$
$$dX=-x_u du/R_1-x_v dv/R_2\ .$$

10. Use the formula

$$\bar{x}=x+r(-ax_u/E^{1/2}-bx_v/G^{1/2}+X)$$

for any point $\bar{x}$ on a line l_1 to show that the differential equation of the Γ_1-curves of the congruence of lines l_1 is

$$T_1du^2+(S_1-P_1)dudv-Q_1\ dv^2=0\ ,$$

and that the foci of the line l_1 are determined by solving

$$(P_1S_1-T_1Q_1)r^2-(S_1+P_1)(EG)^{1/2}r+EG=0\ ,$$

where P_1, Q_1, T_1, S_1 are defined by the formulas

$$P_1=G^{1/2}a_u+E^{1/2}\{21,\ 1\}b+(EG)^{1/2}(1+Da^2)/R_1\ ,$$
$$Q_1=G^{1/2}a_v+E^{1/2}\{22,\ 1\}b+D''ab\ ,$$
$$T_1=E^{1/2}b_u+G^{1/2}\{11,\ 2\}a+Dab\ ,$$
$$S_1=E^{1/2}b_v+G^{1/2}\{12,\ 2\}a+(EG)^{1/2}(1+D''b^2)/R_2\ .$$

Hence show that the Γ_1-curves of the normal congruence are the lines of curvature.

11. The reciprocal line l_2 of the line l_1 of Exercise 10, with respect to any quadric of Darboux, crosses the parametric tangents of the lines of curvature at the points ρ, σ defined by

$$\rho = x + \lambda x_u / E^{1/2}, \qquad \sigma = x + \mu x_v / G^{1/2},$$

where

$$1/\lambda = -a/R_1 + (\log K)_u / 4E^{1/2}, \qquad 1/\mu = -b/R_2 + (\log K)_v / 4G^{1/2}.$$

12. Use the formula

$$\bar{x} = r\rho + (1-r)\sigma$$

for any point $\bar{x}$ on a line l_2 (see Ex. 11) to show that the differential equation of the Γ_2-curves of the congruence of lines l_2 is

$$T_2 du^2 + (S_2 - P_2) du dv - Q_2 dv^2 = 0,$$

and that the foci of the line l_2 are determined by solving

$$Q_2 D^2 G \lambda^2 r^2 + (S_2 - P_2) D D'' (EG)^{1/2} \lambda \mu r (1-r) - T_2 D''^2 E \mu^2 (1-r)^2 = 0,$$

where P_2, Q_2, T_2, S_2 are defined by the formulas

$$\begin{aligned}
P_2 &= D''[E^{1/2}\mu \,(\log \lambda)_u + G^{1/2}\{11, 2\}\lambda + E\mu/\lambda], \\
Q_2 &= D''[E^{1/2}\mu \,(\log \lambda)_v + G^{1/2}\{12, 2\}\lambda + (EG)^{1/2}], \\
T_2 &= D[G^{1/2}\lambda \,(\log \mu)_u + E^{1/2}\{21, 1\}\mu + (EG)^{1/2}], \\
S_2 &= D[G^{1/2}\lambda \,(\log \mu)_v + E^{1/2}\{22, 1\}\mu + G\lambda/\mu].
\end{aligned}$$

For the ray of the lines of curvature $\lambda = -\rho_2$, $\mu = \rho_1$; the ray curves of the lines of curvature on a surface form a conjugate net if, and only if, the surface is isothermic.

13. The formulas for the Weingarten invariants of the lines of curvature are

$$2W^{(u)} = (\log R_1 R_2 D''/D)_{uv}, \qquad 2W^{(v)} = (\log R_1 R_2 D/D'')_{uv}.$$

14. In the notation of Exercise 13 of Chapter IV, the projective normal joins the point x to the point

$$2a x_{vv} + (b - a_u/2a) x_u + (c + a_v/2) x_v$$

when the coordinates x have been multiplied by such a factor that

$$b + 2c' + (\log a^{3/2}/R^2)_u = 0, \qquad 2b' - c/a + (\log a^{1/2}/R^2)_v = 0,$$

where $R = a\mathfrak{B}'^2 + \mathfrak{C}'^2$. LANE, 1927. 9

15. Use the method of Section 49 to show that, when the lines of curvature are parametric, multiplication of the cartesian coordinates x, y, z, 1 by the function λ defined by

$$\lambda^4 K^{1/2} EGD = R^2 D''$$

converts these coordinates into the coordinates that produce the canonical form mentioned in the last exercise for the equations of Exercise 13 in Chapter IV.

16. Use Exercise 15 to show that, in the notation of Section 48, the local direction cosines of the projective normal are proportional to

$$D''E^{1/2}(\log \lambda^2/K^{1/2})_u\,, \qquad DG^{1/2}(\log \lambda^2/K^{1/2})_v\,, \qquad 2DD''\,.$$

17. If an integral surface S of equations (75) with $p=q=0$ is transformed by reciprocal polars, with respect to the sphere $x^2+y^2+z^2=1$, into a surface $\bar{S}$ in such a way that the tangent plane at a point (x, y, z) of S corresponds to the pole $(\bar{x}, \bar{y}, \bar{z})$ of this plane with respect to the sphere, then the coordinates $\bar{x}, \bar{y}, \bar{z}$ satisfy the equations

$$\begin{aligned}
\bar{x}_{uu} &= (a-2b)\bar{x}_u - \beta\bar{x}_v\,.\\
\bar{x}_{uv} &= c\bar{x} - a\bar{x}_u - b\bar{x}_v\,,\\
\bar{x}_{vv} &= -\gamma\bar{x}_u + (\delta-2a)\bar{x}_v\,,
\end{aligned}$$

and we have $K\bar{K}/d^4\bar{d}^4=\text{const.}$ Therefore the property of Tzitzéica is preserved under polar reciprocation with respect to the sphere; this sphere can be replaced by any quadric with the origin as center. TZITZÉICA, 1924. 3, p. 248

18. Every central quadric has the property of Tzitzéica.
TZITZÉICA, 1924. 3, p. 251

19. A ruled surface that is not a quadric and that has the property of Tzitzéica has its two flecnode curves coincident at infinity. TZITZÉICA, 1924. 3, p. 254

20. According to the theorem of Koenigs in Section 35, the point y given by equation (95) with $k\neq 0$ generates a plane net with equal invariants. Find the differential equations of the form (IV, 26) for this net, and prove that it determines a sequence of Laplace of period three. WILCZYNSKI, 1914. 3, p. 141

21. The quadric (55) is the quadric of Lie at a point of a surface in case k_2, k_3 have the values given in equations (60) and k_4 has the value given by

$$\begin{aligned}
4k_4 = 2(k_2^2R_1+k_3^2R_2) &+ (1/R_1+1/R_2)(1+k_2^2R_1^2\sin^2 A+k_3^2R_2^2\cos^2 A)\\
&+(k_2R_1\sin A)_u/E^{1/2}\sin A+(k_3R_2\cos A)_v/G^{1/2}\cos A\,,
\end{aligned}$$

where A is an angle such that

$$(R_2-R_1)\sin^2 A = R_2\,, \qquad (R_1-R_2)\cos^2 A = R_1\,.$$

DEMOULIN, 1908. 4, p. 566

22. The centers of all the quadrics of Darboux at a point $P(0, 0, 0)$ of a surface S lie on the straight line

$$\zeta = -\xi/R_1k_2 = -\eta/R_2k_3 ,$$

where k_2, k_3 are given by equations (60). This is the line of intersection of the two asymptotic tangent planes through P of the two non-developable ruled surfaces of asymptotic tangents circumscribing S along the two asymptotic curves through P.

23. Investigate the developables and focal surfaces of the congruence generated by the line of centers of the quadrics of Darboux at a point P on a surface S (see Ex. 22). Prove that the developables intersect S in a conjugate net, and that the foci of the generator separate harmonically the point P and the center of the quadric of Lie of S at P. DEMOULIN, 1908. 4

24. The equations of the transformation of coordinates from the local tetrahedron x, x_{-1}, x_1, y in the situation of Section 30 to the local trihedron of Section 48 are

$$\begin{aligned}
x_1 &= 1+\xi/\rho_2-\eta/\rho_1+(\zeta/2)[(R_1+R_2+R_2\rho_{1v}/G^{1/2})/\rho_1^2+(R_2+R_1-R_1\rho_{2u}/E^{1/2})/\rho_2^2] ,\\
x_2 &= (\xi+R_2\zeta/\rho_2)/E^{1/2} ,\\
x_3 &= (\eta-R_1\zeta/\rho_1)/G^{1/2} ,\\
x_4 &= \zeta .
\end{aligned}$$

CHAPTER VII

SURFACES AND VARIETIES

Introduction. This chapter contains some further developments of the projective differential geometry of surfaces in hyperspace. Moreover, there are certain portions of the theory of varieties in general which are so intimately connected with the theory of surfaces that it seems appropriate to include them here also. Varieties are to be regarded as generalizations of curves and surfaces, a curve being a variety of one dimension, while a surface is a variety of two dimensions.

The sections of this chapter fall naturally into two groups, the first group including the first three sections and finding its inspiration in the researches of Segre. Section 52 is devoted to a study of the neighborhoods of various orders of a point on a surface, or on a variety, in hyperspace. Section 53 contains an investigation of the different forms assumed, in the neighborhood of a point on a surface, by the curve of intersection of the surface and a variable hyperplane through the point. Surfaces in space of five dimensions are of sufficient interest to occupy the next section.

In the next group, Section 55 is an introduction to the fundamentals of Segre's monumental work on varieties which are the loci of linear spaces. An example of such a variety, also studied by Segre, is discussed in the following section, namely, the locus of the tangent planes of a surface. Section 57 is taken up with a consideration of sets of varieties which are the loci of linear spaces with the generating linear spaces in correspondence. The developments in this section are mainly due to the author.

Frequent use will be made of the intersection formula

$$V_h^p V_k^q = V_{h+k-n}^{pq} .$$

This formula* is to be interpreted as stating that two varieties of orders p, q respectively and of dimensions h, k in a space S_n intersect in a variety of order pq and dimensions $h+k-n$, when $h+k \geqq n$. We shall explain the notion of *order of a variety* in the next section. We may state in passing that we shall only speak of *order* for an algebraic variety, that is, a variety that can be defined as the locus of a point whose coordinates satisfy one or more algebraic equations.

52. The neighborhoods of a point on a surface, or on a variety. The primary purpose of this section is to define and study systematically the neigh-

* Bertini, 1923. 2, p. 254.

borhoods of various orders of an ordinary point on an analytic surface. The neighborhoods of the first, second, and third orders will be considered in some detail. A local coordinate system will be introduced for each of these neighborhoods, and the equations will be written for the more interesting configurations associated with the point of the surface in each case. Among these *the cone of Del Pezzo* is perhaps the most worthy of special mention.

Although a complete theory of analytic varieties is beyond the scope of this book, it nevertheless seems desirable at the end of this section to extend to an analytic variety of m dimensions certain of the definitions made and results obtained for a surface. A noteworthy result is the formula due to Mendel for the dimensionality of the space $S(k, r)$ at a point of a variety V_m in space S_n.

Let us consider, in a linear space of n dimensions S_n, a proper analytic surface S whose parametric vector equation is $x=x(u, v)$. The neighborhood of the first order of an ordinary point P_x on the surface S is determined geometrically by P_x and one point consecutive to P_x in each direction from P_x; this neighborhood consists of the point P_x and the configurations that can be associated with P_x analytically by means of x and the two first partial derivatives x_u, x_v. Similarly, the neighborhood of the second order of the point P_x consists of P_x and the configurations that can be associated with P_x by means of the first and second partial derivatives of x with respect to u and v. In general, *the neighborhood of order k of a point P_x on a surface consists of P_x and the configurations that can be associated with P_x by means of the partial derivatives of x with respect to u and v up to and including those of order k.*

Through each point P_x of a surface there is a curve C of the family defined by the differential equation $dv-\lambda du=0$. It will be useful to calculate some of the total derivatives of x along the curve C. We find

$$(1)\qquad \begin{cases} x'=x_u+x_v\lambda & (x'=dx/du)\ , \\ x''=G+x_v\lambda'\ , \\ x'''=H+3(x_{uv}+x_{vv}\lambda)\lambda'+x_v\lambda''\ , \\ x^{\text{iv}}=I+6J\lambda'+3x_{vv}\lambda'^2+4(x_{uv}+x_{vv}\lambda)\lambda''+x_v\lambda'''\ , \end{cases}$$

where G, H, I, J are defined by placing

$$(2)\qquad \begin{cases} G=x_{uu}+2x_{uv}\lambda+x_{vv}\lambda^2\ , \\ H=x_{uuu}+3x_{uuv}\lambda+3x_{uvv}\lambda^2+x_{vvv}\lambda^3\ , \\ I=x_{uuuu}+4x_{uuuv}\lambda+6x_{uuvv}\lambda^2+4x_{uvvv}\lambda^3+x_{vvvv}\lambda^4\ , \\ J=x_{uuv}+2x_{uvv}\lambda+x_{vvv}\lambda^2\ . \end{cases}$$

We restate here for convenience the definition (see § 10) of the space $S(k, r)$. *The osculating space* $S(k, r)$ with respect to an element E_r (see § 3) of a curve at a point P_x of a surface S is the ambient of, or linear space of least dimensions containing, the osculating space S_k at the point P_x of every curve on the surface S that passes through P_x and has at P_x the same element E_r $(r<k)$. In particular, *the osculating space* $S(k, 0)$ at a point P_x of a surface S is the ambient of the osculating space S_k at P_x of every curve on the surface S that passes through P_x. It follows that *the space* $S(k, 0)$ *is the linear space determined by* x *and the partial derivatives of* x *up to and including those of order* k. The formula for the number of these derivatives shows that $S(k, 0)$ *is ordinarily a linear space of dimensions* $k(k+3)/2$. Moreover, the neighborhood of order k of a point on a surface lies in the space $S(k, 0)$ at the point.

The space $S(1, 0)$ *is the tangent plane* at a point P_x of a surface S; the theory of *the neighborhood of the first order* in this plane is already familiar to the reader. The tangent line of a curve C at the point P_x is determined by x, x'. The locus of the tangent lines at the point P_x of all curves on the surface S that pass through P_x is a flat pencil with center at P_x and lying in the tangent plane. If a local coordinate system is introduced in this plane, for which the triangle of reference has the vertices x, x_u, x_v and for which the unit point is chosen so that the point $x_1x+x_2x_u+x_3x_v$ has local coordinates proportional to x_1, x_2, x_3, then the equation of the tangent line of the curve C is

$$\lambda x_2 - x_3 = 0 . \tag{3}$$

*The neighborhood of the second order** of the point P_x lies in the space $S(2, 0)$, which is ordinarily a space S_5. The osculating plane of the curve C at the point P_x is determined by x, x', x''. *The space* $S(2, 1)$ *of the surface* S *in the direction of the curve* C *at the point* P_x is studied by holding λ fixed and allowing λ' to vary in the formulas for x', x'' in the first two of equations (1). Under these circumstances x and x' are fixed, but the point x'' describes the straight line joining the fixed point x_v to the fixed point G (see Fig. 40). Consequently the osculating plane describes an axial pencil with the tangent of the curve C as axis and lying in the space S_3 of the points

$$x, \qquad x_u, \qquad x_v, \qquad G . \tag{4}$$

This space S_3 is the space $S(2, 1)$ under consideration.

When the curve C varies through the point P_x so that its tangent line at P_x assumes all possible directions in the tangent plane, then λ varies and

* Segre, 1907. 2, p. 1050.

the space $S(2, 1)$ varies, joining always the fixed tangent plane to the variable point G, and generating a variety of four dimensions V_4. Calling the locus of a one-parameter family of spaces S_h through a fixed space $S_k(h>k)$ *a cone* with the spaces S_h for generators and with the space S_k for vertex, we see that in this extended sense of the word the variety V_4 is a cone whose vertex is the tangent plane and whose generators are the spaces $S(2, 1)$.

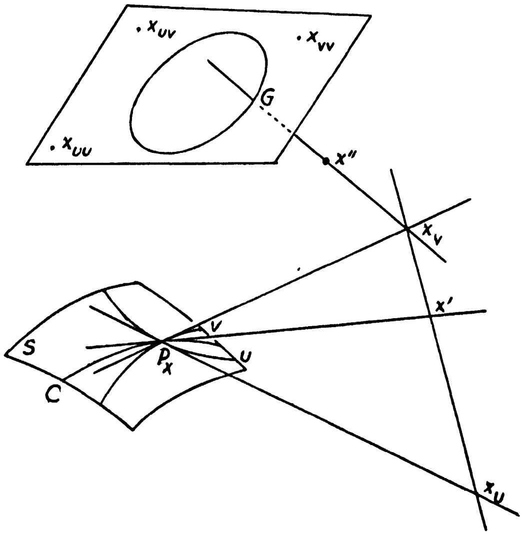

FIG. 40

This cone is called *the cone* of Del Pezzo* at the point P_x of the surface S. It can also be characterized as *the locus of the* ∞^2 *osculating planes at the point* P_x *of all the curves on the surface* S *that pass through* P_x.

If a local coordinate system is introduced in the space $S(2, 0)$ so that a point $x_1x+ \cdots +x_6x_{vv}$ has local coordinates proportional to $x_1, \ldots, x_6$, the equations of the configurations in the neighborhood of the second order can be calculated. The equations of the space $S(2, 1)$ can be written in the form

$$2\lambda x_4 - x_5 = 0\ , \qquad \lambda x_5 - 2x_6 = 0\ , \tag{5}$$

and elimination of λ therefrom gives *the equation of the cone of Del Pezzo,*

$$x_5^2 - 4x_4x_6 = 0\ . \tag{6}$$

* Del Pezzo, 1886. 1, p. 177.

The equations of *the locus of the point* G are

$$x_1=x_2=x_3=x_5^2-4x_4x_6=0\ . \tag{7}$$

Therefore this locus is a conic in the plane of the points x_{uu}, x_{uv}, x_{vv}, the equations of this plane being

$$x_1=x_2=x_3=0\ . \tag{8}$$

The neighborhood of the third order of the point P_x lies in the space $S(3, 0)$, which is ordinarily a space S_9. The first problem is to study *the space* $S(3, 2)$ associated with a curve C through the point P_x on the surface S. The osculating space S_3 of the curve C at the point P_x is determined by x, x', x'', x'''. When λ, λ' are fixed and λ'' varies in the first three of equations (1), the points x, x', x'' are fixed and the point x''' varies on the line joining the point x_v to the fixed point $H+3(x_{uv}+x_{vv}\lambda)\lambda'$. Calling the locus of a linear one-parameter family of spaces S_{k-1} through a fixed space S_{k-2} in a space S_k *a pencil* with the space S_{k-2} for base, we see that, under the conditions stated, the osculating space S_3 of the curve C varies and has for its locus a pencil with the osculating plane as base and lying in the space S_4 of the points

$$x,\qquad x_u,\qquad x_v,\qquad G,\qquad H+3(x_{uv}+x_{vv}\lambda)\lambda'\ . \tag{9}$$

This space S_4 is the space $S(3, 2)$ under consideration; it contains the space $S(2, 1)$ previously studied.

The next problem is to investigate *the space* $S(3, 1)$ in the direction of the curve C. This problem is the same as to investigate the locus of the space $S(3, 2)$ when λ' varies and λ is fixed. Under these circumstances the space $S(3, 2)$ varies and has for its locus a pencil with the space $S(2, 1)$ as base and lying in the space S_5 of the points

$$x,\qquad x_u,\qquad x_v,\qquad x_{uu}+x_{uv}\lambda,\qquad x_{uv}+x_{vv}\lambda,\qquad H\ . \tag{10}$$

This space S_5 is the space $S(3, 1)$ under consideration, and contains the tangent plane of the surface S at the point P_x.

Finally, the locus of the space $S(3, 1)$ when the curve C varies through the point P_x, so that its tangent line at P_x assumes all possible directions in the tangent plane, is a variety of six dimensions V_6. This variety is a cone whose vertex is the tangent plane and whose generators are the spaces $S(3, 1)$.

The order of an algebraic variety V_m of dimensions m in space S_n is defined to be the number of points in which it is met by any linear space S_{n-m}

which is not a component of the variety V_m. In discussing the order of a cone the linear space whose intersection with the cone is used must have no point in common with the vertex of the cone. So, for example, *the cone of Del Pezzo is of order two,* since the plane of the points x_{uu}, x_{uv}, x_{vv} meets it in a conic, which is certainly of order two. The symbol for the cone of Del Pezzo is V_4^2, the subscript indicating the dimensions and the superscript the order of the cone.

For the purpose of showing that *the cone V_6 defined above is of order five,* so that it may be denoted by the symbol V_6^5, let us observe that the order of this cone is the order of the locus of the plane of the last three of the points (10) when λ varies. This locus is a variety V_3 in the space S_6 of the points $x_{uu}, \ldots, x_{vvv}$. Now, when λ varies, the locus of the point $x_{uu}+x_{uv}\lambda$ is obviously the straight line joining the points x_{uu}, x_{uv}; similarly, the locus of the point $x_{uv}+x_{vv}\lambda$ is a straight line; and the locus of the point H is a twisted cubic curve in the space S_3 of the points $x_{uuu}, \ldots, x_{vvv}$. Moreover, the three points where the variable plane meets the two straight lines and the cubic curve are projectively related, all three points corresponding to the same value of λ. Therefore the order of the locus of the plane is five, being the sum of the orders of the director curves.

For some purposes it is advantageous to introduce a local coordinate system in the space $S(3, 0)$ so that a point $x_1x+\cdots+x_{10}x_{vvv}$ has local coordinates proportional to $x_1, \ldots, x_{10}$. Then the equations of the space $S(3, 2)$ can be written in the form

$$(11) \qquad \begin{cases} 2\lambda x_4-x_5+3\lambda' x_7=0\,, \qquad \lambda^2x_4-x_6+3\lambda\lambda' x_7=0\,, \\ x_8-3\lambda x_7=0\,, \qquad x_9-3\lambda^2x_7=0\,, \qquad x_{10}-\lambda^3x_7=0\,. \end{cases}$$

The equations of the space $S(3, 1)$ are found, by eliminating λ' from equations (11), to be

$$(12) \qquad \begin{cases} \lambda^2x_4-\lambda x_5+x_6=0\,, \qquad x_8-3\lambda x_7=0\,, \\ \qquad x_9-3\lambda^2x_7=0\,, \qquad x_{10}-\lambda^3x_7=0\,. \end{cases}$$

Finally, the equations of the cone V_6^5 are found, by eliminating λ from equations (12), to be

$$(13) \qquad x_8^2-3x_7x_9=0\,, \qquad x_9^2-3x_8x_{10}=0\,, \qquad x_4x_9-x_5x_8+3x_6x_7=0\,.$$

It may be remarked that the cone V_6^5 is only part of the intersection of the three hyperquadrics whose equations appear here, the space S_6 with the equations $x_9=x_8=x_6=0$, and the space S_6 with the equations $x_9=x_8=x_7=0$ *counted twice,* being excluded.

We shall not prolong further the detailed discussion of particular neighborhoods of a point P_x on a surface S, but shall state without proof* two or three results concerning the general neighborhood of order k. For this purpose let us denote by $V(k, r)$ *the locus* of the osculating space S_k at the point P_x of every curve on the surface S through P_x having at P_x the same element E_r. *The space* $V(k, r)$ *is of dimensions* $2k-r$. *The space* $S(k, r)$ *is of dimensions*

$$[k(k+r+3)+c(r-1-c)]/2(r+1) \ , \tag{14}$$

where

$$0 \leqq c \leqq r \ , \qquad k \equiv c \ (\text{mod } r+1) \ .$$

The space $V(k, r)$ *is linear when* $r \geqq (k-1)/2$.

Although this book is not primarily concerned with varieties in general, it seems well to define a general analytic variety and to state a few results concerning the general neighborhood of a point thereon. *An analytic variety* V_m *in space* S_n *is defined to be the locus of a point whose coordinates* x *are analytic functions of* m *(and not fewer than* m*) independent parameters* $u^1, \ldots, u^m$ $(m \leqq n)$. The parametric vector equation of the variety V_m can be written in the form

$$x = x(u^1 \, , \ldots , u^m) \ . \tag{15}$$

The tangent linear space S_m *at a point* P_x *of a variety* V_m *is defined to be the ambient of the tangent line at* P_x *of every curve on* V_m *that passes through* P_x. Such a curve can be represented by equations of the form

$$u^i = u^i(t) \qquad\qquad (i = 1, \ldots , m) \ ,$$

where t is an independent variable. The tangent space S_m is easily shown to be determined by the points $x, x_1, \ldots, x_m$, subscripts indicating partial differentiation with respect to the corresponding variables $u^1, \ldots, u^m$. This space S_m is the space $S(1, 0)$ at the point P_x of the variety V_m.

The definitions of the spaces $S(k, 0)$, $V(k, r)$, $S(k, r)$ as made in connection with surfaces can easily be restated so as to be available in the theory of a general variety V_m. We shall not enter into details but shall state a few results without proof. *The space* $S(k, 0)$ *is of dimensions* $C_{m+k,\,k}-1$, the number of combinations of $m+k$ things taken k at a time being denoted by

* Mendel, 1930. 3. The proof may be found in Mendel's Chicago doctoral dissertation, 1930.

$C_{k+m,\,k}$. *The space* $V(k, r)$ *is of dimensions* $m(k-r)+r$. *The space* $S(k, r)$ *is** *of dimensions*

$$(r+1)C_{m+p-1,\,m}+(c+1)C_{m+p-1,\,p}-1\ , \tag{16}$$

where

$$0\overline{\gtrless} c\overline{\gtrless} r\ ,\qquad k=p(r+1)+c\ ,$$

and k, r, c, p *are positive integers or zero.*

53. Hyperplane sections of a surface. The intersection of a surface and a hyperplane is a curve. A variable hyperplane through a fixed point of a surface intersects the surface in a curve which may assume various forms in the neighborhood of the point according to the different possible relative positions of the surface and hyperplane. Some of the possibilities as to the nature of the curve will now be considered, the surface being unspecialized. The investigation will be continued in the special case in which the surface is an integral surface of a differential equation of the second order.

The equation of a hyperplane S_{n-1} in a space S_n can be written with customary abbreviation in the form $\Sigma\xi x=0$, the summation ranging from 1 to $n+1$. The $n+1$ coefficients ξ are homogeneous coordinates of the hyperplane. Let us consider a surface S with the parametric vector equation $x=x(u, v)$ in the space S_n. The coordinates X of a point near an ordinary point P_x on the surface S can be represented by Taylor's expansion as power series in the increments Δu, Δv corresponding to displacement on S from P_x to the point X. If the hyperplane ξ passes through the point X, we have $\Sigma\xi X=0$ and consequently

$$\Sigma\xi x+\Sigma\xi x_1\Delta u+\Sigma\xi x_2\Delta v+(\Sigma\xi x_{11}\Delta u^2+\cdots)/2+\cdots=0\ , \tag{17}$$

the subscripts 1, 2 now denoting partial derivatives with respect to u, v respectively. If the hyperplane ξ passes through the point P_x, then $\Sigma\xi x=0$. Such a hyperplane ordinarily intersects the surface S in a curve with a simple point at P_x; the direction of this curve at P_x is found, by means of equation (17), to be given by

$$\Sigma\xi x_1du+\Sigma\xi x_2dv=0\ . \tag{18}$$

A hyperplane ξ which contains the tangent plane at the point P_x of the surface S has coordinates which satisfy the conditions

$$\Sigma\xi x=\Sigma\xi x_1=\Sigma\xi x_2=0\ . \tag{19}$$

* *Ibid.* For the proof see Mendel's dissertation previously cited.

Such a hyperplane ordinarily intersects the surface S in a curve with a double point at P_x; the directions of the double-point tangents are given by

$$\Sigma\xi x_{11}du^2+2\Sigma\xi x_{12}dudv+\Sigma\xi x_{22}dv^2=0 \ . \tag{20}$$

The curve of section of the surface S made by a hyperplane through the tangent plane has two coincident double-point tangents at the point P_x if, and only if, the coordinates ξ of the hyperplane satisfy not only equations (19) but also

$$\Sigma\xi x_{11}\Sigma\xi x_{22}-(\Sigma\xi x_{12})^2=0 \ . \tag{21}$$

The double point is then ordinarily a cusp. In this case the direction of the cusp tangent satisfies the two equivalent equations

$$\Sigma\xi x_{11}du+\Sigma\xi x_{12}dv=0 \ , \qquad \Sigma\xi x_{12}du+\Sigma\xi x_{22}dv=0 \ . \tag{22}$$

It is interesting in this connection to consider the equations of certain already familiar configurations in hyperplane coordinates. A conic meets a hyperplane in two points. Replacing λ by dv/du in the first of equations (2) and calculating the expression $\Sigma\xi G$, we see that the conic which is the locus of the point G as λ varies meets any hyperplane ξ in two points which are obtained by solving equation (20) for the ratio dv/du. These two points coincide if, and only if, equation (21) is satisfied. Then the hyperplane contains a tangent of the conic and may be said to be itself tangent to the conic. Thus (21) *is the equation of the said conic in hyperplane coordinates.* Moreover, *equations* (19), (21) *taken together are the equations of the cone of Del Pezzo in hyperplane coordinates.*

A hyperplane ξ which contains the space $S(2, 0)$ at the point P_x of the surface S has coordinates which satisfy not only the conditions (19) but also

$$\Sigma\xi x_{11}=\Sigma\xi x_{12}=\Sigma\xi x_{22}=0 \ . \tag{23}$$

Such a hyperplane ordinarily cuts S in a curve with a triple point at P_x; the directions of the triple-point tangents are given by

$$\Sigma\xi x_{111}du^3+3\Sigma\xi x_{112}du^2dv+3\Sigma\xi x_{122}dudv^2+\Sigma\xi x_{222}dv^3=0 \ . \tag{24}$$

Various possibilities can arise with regard to the coincidences of these tangents, but we shall not go into the details here. Moreover, we shall leave to the reader the discussion of the curve of intersection of a surface

and a hyperplane through the space $S(k, 0)$ at a point of the surface with $k>2$, and with k general.

An integral surface of just one second-order differential equation of the form (IV, 10) *will be called for brevity a surface F.* Such a surface has many special properties, some of which have already been observed in Chapter IV. For example, such a surface sustains a conjugate net or else a one-parameter family of asymptotic curves. Moreover, the space $S(2, 0)$ at a point P_x of a surface F is a space S_4.

There are some special properties of a surface F which are connected with the hyperplane sections of the surface. For example, *the pairs of double-point tangents, at a point P_x on a surface F, of the curves of section of F and the hyperplanes containing the tangent plane of F at P_x are the ∞^1 pairs of rays in the involution** *having the conjugate tangents (supposed distinct) at P_x for self-corresponding rays.* In order to demonstrate the truth of this statement let us observe that the directions of the double-point tangents are given by equation (20), and the directions of the conjugate tangents by

$$Cdu^2-2Bdudv+Adv^2=0\,. \tag{25}$$

Moreover, equations (19) and (IV, 10) imply

$$A\Sigma\xi x_{11}+2B\Sigma\xi x_{12}+C\Sigma\xi x_{22}=0\,. \tag{26}$$

But this equation says that the harmonic invariant of the binary quadratic forms in equations (20) and (25) vanishes. Thus the proof is completed.

It will next be shown that *the conjugate directions at a point of a surface F are the hessian pair of the triple of directions given by equation* (24). Use will be made of the property of the hessian pair of a triple which is set forth in Exercise 6. Let us for convenience write equations (19), (23) together in the concise form

$$\Sigma\xi x=0\,,\qquad \Sigma\xi x_p=0\,,\qquad \Sigma\xi x_{pq}=0 \qquad (p,\, q=1,\, 2)\,. \tag{27}$$

It is easy to deduce from (27) and the equations obtained therefrom by differentiation the relations

$$\Sigma x\xi=0\,,\qquad \Sigma x\xi_p=0\,,\qquad \Sigma x\xi_{pq}=0\,,\qquad \Sigma x_p\xi_q=0\,. \tag{28}$$

Moreover, continuation of the calculations leads to

$$\Sigma\xi_r x_{pq}=-\Sigma\xi x_{rpq}=\Sigma x\xi_{rpq}=-\Sigma x_r\xi_{pq} \qquad (p,\, q,\, r=1,\, 2)\,. \tag{29}$$

* Segre, 1907. 2, p. 1063.

From equation (IV, 10) one now obtains

$$A\Sigma\xi_p x_{11}+2B\Sigma\xi_p x_{12}+C\Sigma\xi_p x_{22}=0 \qquad (p=1,\,2)\,,$$

and consequently

$$A\Sigma\xi x_{11p}+2B\Sigma\xi x_{12p}+C\Sigma\xi x_{22p}=0 \qquad (p=1,\,2)\,. \tag{30}$$

The equation of the first polar pair of an arbitrary direction $\delta v/\delta u$ with respect to the triple of directions (24) is

$$\left\{\begin{aligned}&\delta u(\Sigma\xi x_{111}du^2+2\Sigma\xi x_{121}dudv+\Sigma\xi x_{221}dv^2)\\&\qquad\qquad+\delta v(\Sigma\xi x_{112}du^2+2\Sigma\xi x_{122}dudv+\Sigma\xi x_{222}dv^2)=0\,.\end{aligned}\right. \tag{31}$$

The demonstration is completed by observing that equation (30) says that the harmonic invariant of the binary quadratic form in equation (31), in the differentials du, dv, and of the form in equation (25) in the same variables vanishes identically in δu, δv.

It is clear geometrically that *the conjugate tangents at a point of a surface F are the cusp tangents of the curves of section made by those hyperplanes through the point that cut the surface F in curves with cusps at the point.* Moreover, at a point of a surface F, the hyperplane envelope represented by equation (21) breaks up into two spaces S_3 within the space $S(2,\,0)$, as can be verified by use of equations (21), (26). The sections with cusps are made by the hyperplanes through one or the other of these two spaces S_3.

54. Surfaces immersed in space S_5. There are certain special properties of surfaces immersed in space S_5 which are of considerable interest. For instance, there are on such a surface five noteworthy covariant one-parameter families of curves called *principal curves.* In this section these curves will be considered first on the most general analytic surface and then on a surface F in space S_5. Investigation of those surfaces on which the principal curves are indeterminate will introduce *the surface of Veronese.*

Let us consider some of the possible relative positions of a hyperplane S_4 and a surface in space S_5. There are in all ∞^5 hyperplanes in space S_5. Among them there are ∞^2 hyperplanes containing the tangent plane at a point P_x of a surface S; the coordinates ξ of any one of the latter satisfy the three equations (19). Among these latter there are ∞^1 hyperplanes that cut the surface S in curves with cusps at the point P_x; the coordinates of any one of these ∞^1 hyperplanes satisfy also equation (21), or what is the same thing, the two equations (22). It is easy to show that among these hyperplanes there are just five that cut the surface S in curves with *tacnodes* at the point P_x. In fact, the coordinates of any such hyperplane must satisfy

also equation (24). Elimination of the six homogeneous coordinates ξ from the six equations (19), (22), (24) leads to the equation

$$(32)\quad \begin{cases} (x,\ x_1,\ x_2,\ x_{11}du+x_{12}dv,\ x_{12}du+x_{22}dv, \\ \qquad\qquad x_{111}du^3+3x_{112}du^2dv+3x_{122}dudv^2+x_{222}dv^3)=0\ . \end{cases}$$

This equation is of the fifth degree in the ratio dv/du, and consequently *there are five hyperplanes which produce tacnodal sections*, as stated.

We now state some definitions. *The five directions of tacnodal section at a point of a surface in space S_5 are called the principal directions at the point, and the tangents in these directions are called principal tangents. The five one-parameter families of curves enveloped by the principal tangents are called the principal curves on the surface.* Equation (32) may be regarded as the curvilinear differential equation of the principal curves.

The coordinates x of a general point on a surface which is non-developable and not a surface F and is immersed in space S_5 satisfy* a system of four equations expressing each of the third derivatives of x as a linear combination of x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}. If non-homogeneous coordinates are used, and if two of them are taken as the independent variables u, v, the system of differential equations can be written in the form

$$(33)\qquad \begin{cases} x_{uuu}=ax_{uu}+rx_{uv}+\delta x_{vv}\ , \\ x_{uuv}=bx_{uu}+sx_{uv}+\gamma x_{vv}\ , \\ x_{uvv}=cx_{uu}+\sigma x_{uv}+\beta x_{vv}\ , \\ x_{vvv}=dx_{uu}+\rho x_{uv}+\alpha x_{vv}\ . \end{cases}$$

In this notation the curvilinear differential equation (32) of the principal curves becomes

$$(34)\quad \begin{cases} \delta du^5+(3\gamma-r)du^4dv+(a-3s+3\beta)du^3dv^2 \\ \qquad\qquad +(3b-3\sigma+\alpha)du^2dv^3+(3c-\rho)dudv^4+ddv^5=0\ . \end{cases}$$

In space S_5 integral surfaces F of a differential equation of the form (IV, 10) are of particular interest. For example, one can easily demonstrate the following theorem. *Through each point P_x of a surface F in space S_5 there is a unique hyperplane S_4 which cuts F in a curve with a triple point at P_x; this hyperplane is the space $S(2,\ 0)$ at P_x.* For the demonstration one observes that the six coordinates ξ of any hyperplane cutting the surface F in a curve with a triple point at a point P_x satisfy the six equations (19), (23). But

* Beenken, 1928. 6, p. 2.

just five of these equations are independent in the present situation, since account must be taken of equation (26). These five equations can be solved uniquely for the ratios of the coordinates ξ, and then the unique hyperplane is observed to be the space $S(2, 0)$, as asserted. The directions of the triple-point tangents of the curve of section of the surface F made by the space $S(2, 0)$ at the point P_x are given by equation (24).

It will now be shown that *the five principal directions at a point P_x of a surface F in space S_5 consist* of the two conjugate directions and the three directions of the curve of intersection of the surface F and the space $S(2, 0)$ at the point P_x.* In addition to the equations previously used in studying the principal directions, equation (26) is now valid. There are two possibilities with respect to equations (26) and (22) regarded as simultaneous linear homogeneous algebraic equations in the expressions $\Sigma\xi x_{11}$, $\Sigma\xi x_{12}$, $\Sigma\xi x_{22}$. First, it may be that the hyperplane is such that these expressions are not all zero. Then the determinant of their coefficients must vanish. The vanishing of this determinant leads to equation (25). Therefore *the conjugate directions are two of the principal directions.* Second, if the expressions considered are all zero, equations (23) being valid, the hyperplane is the space $S(2, 0)$ at the point P_x, and the principal directions are given by equation (24). This completes the proof.

The question arises as to what surfaces are such that on them the principal curves are indeterminate. It can easily be shown that *the principal curves are not indeterminate on a surface F immersed in space S_5.* For, in the first place, since not all of A, B, C are zero, the conjugate net, or the family of asymptotics as the case may be, cannot be indeterminate. In the second place, it can be shown by an indirect proof that the curves defined on a surface F by equations (24) cannot be indeterminate. For, if they were, then we should have

$$\Sigma\xi x_{pqr}=0 \qquad (p,\, q,\, r=1,\, 2)\,. \tag{35}$$

If the surface F sustains a conjugate net, and if this net is taken as parametric, then the coordinates ξ of the hyperplane $S(2, 0)$ can be written in the form

$$\xi=(x,\, x_1,\, x_2,\, x_{11},\, x_{22})\,. \tag{36}$$

Equations (35), (36) imply

$$(x,\, x_1,\, x_2,\, x_{11},\, x_{22},\, x_{pqr})=0\,. \tag{37}$$

* Segre, 1921. 1, p. 203.

Therefore each third derivative of x can be expressed as a linear combination of $x, x_1, x_2, x_{11}, x_{22}$, and the surface F is in a space S_4. Thus the assumption that the surface F is immersed in space S_5 is contradicted. If instead the surface F sustains a family of asymptotics which are taken as the v-curves, then the coordinates ξ of the hyperplane $S(2, 0)$ can be written in the form

$$\xi = (x, x_1, x_2, x_{11}, x_{12}) , \tag{38}$$

and the same contradiction can be reached.

It can be shown that *the principal curves are indeterminate on a developable surface.* For, according to Exercise 3 of Chapter IV, the coordinates x of a general point on a developable surface satisfy two independent equations of the form (IV, 10). Then it is possible to eliminate two of the second derivatives from equation (32); consequently the first five columns of the determinant in (32) are linearly dependent; hence equation (32) is an identity in dv, du.

We now state a definition. *A surface of Veronese is a surface* immersed in space S_5 and sustaining a two-parameter family of conics.* The algebraic equations of a surface of Veronese can be written in the form

$$x_1x_5 = x_4x_6 , \qquad x_2x_6 = x_4x_5 , \qquad x_3x_4 = x_5x_6 , \tag{39}$$

and a parametric representation of this surface is

$$x_1 = u^2 , \qquad x_2 = 1 , \qquad x_3 = v^2 , \qquad x_4 = u , \qquad x_5 = v , \qquad x_6 = uv . \tag{40}$$

Since these coordinates x satisfy the differential equations

$$x_{pqr} = 0 \qquad (p, q, r = 1, 2) , \tag{41}$$

it follows that *the principal curves are indeterminate on a surface of Veronese* (see Exs. 18, 19).

55. Varieties which are the loci of linear spaces. In this and the next two sections attention will be focused on varieties† which are the loci of linear spaces. In the present section a variety V_{k+1} which is the locus of ∞^1 linear spaces S_k in space S_n is briefly studied. The most interesting special case is perhaps the case $k=1$, in which a variety V_{k+1} is a ruled surface. Some of the results obtained for a variety V_{k+1} are employed in a generalization

* Bertini, 1923. 2, p. 394.

† Segre, 1910. 1.

of the theory, in which one considers a variety V_{k+m} which is the locus of ∞^m linear spaces S_k.

The word "consecutive" is unfortunately used in two senses in differential geometry. The first is connected with *a limiting process* and is the sense in which the word has been used hitherto in this book. In this sense of the word the usual definition of the tangent at an ordinary point of a curve is succinctly stated by saying that the tangent is the line that joins the point to a consecutive point on the curve. Moreover, three consecutive points determine an osculating plane of a curve, and four consecutive points, or else two consecutive tangent lines, determine an osculating space S_3 of the curve. Finally, according to Exercise 1, two consecutive tangent planes of a surface determine a space S_5. The second sense of the word is connected with *the order of an infinitesimal,* and when the word is used in this sense in the following pages it will be italicized. According to this sense, two *consecutive* generators of a developable surface intersect in a point, so that two *consecutive* tangents of a curve determine an osculating plane of the curve. Moreover, two *consecutive* tangent planes of a surface have a point in common and hence determine a space S_4.

A variety V_{k+1} which is the locus of ∞^1 linear spaces S_k in space $S_n (k < n-1)$ can be represented analytically in the following way. A space S_k can be regarded as determined by $k+1$ linearly independent points $x_1, \ldots, x_{k+1}$ with the coordinates

$$x_i^j \qquad (i=1, \ldots, k+1; j=1, \ldots, n+1) . \tag{42}$$

Let all the coordinates of all the $k+1$ points be single-valued analytic functions of the same independent variable t. Then, as t varies, the locus of each point x_i is a curve C_i, and the locus of the space S_k is a variety V_{k+1}. Any point y on this variety is given by the equation

$$y = \Sigma u_i x_i \qquad (i=1, \ldots, k+1) , \tag{43}$$

in which the coefficients $u_1, \ldots, u_{k+1}$ are parameters which are independent of t. Therefore *equation* (43) *is the parametric vector equation of the variety* V_{k+1}.

The curves C_i may be called *director curves* of the variety V_{k+1}. The points x_i of these curves are in correspondence, a set of corresponding points consisting of those that correspond to a particular value of the parameter t. Each set of corresponding points determines a generator S_k, and on this generator the parameters u_i are projective homogeneous coordinates of the point y.

Any curve C through a point P_y on a variety V_{k+1} may be represented by equations of the form

$$(44) \qquad u_i = u_i(t) \qquad (i=1, \ldots, k+1),$$

in which t is an independent variable. The tangent line of the curve C at the point P_y is determined by P_y and the point y' given by

$$(45) \qquad y' = \Sigma u'x + z,$$

where the accents denote differentiation with respect to t, and z is defined by

$$(46) \qquad z = \Sigma ux' .$$

The tangent space S_{k+1} at a point P_y of a variety V_{k+1} is by definition the ambient of the tangent line at P_y of every curve on V_{k+1} that passes through P_y. This tangent space can be shown to be determined by the points

$$(47) \qquad x_1, \ldots, \quad x_{k+1}, \quad z,$$

either by applying the result in the latter part of Section 52 on the determination of the tangent space S_m at a point of a variety V_m, or else by observing that when the point P_y is fixed then t and u_i are fixed and that when the curve C through P_y varies in all possible ways then the derivatives u_i' vary, so that the point y' varies in the space S_{k+1}.

The tangent space S_{2k+1} along a generator S_k of a variety V_{k+1} is defined to be the ambient of the tangent space S_{k+1} at a point P_y on the generator S_k as P_y varies over S_k. Holding t fixed and allowing u_i to vary, we see that the point z describes the space of the points $x_1', \ldots, x_{k+1}'$. Therefore the tangent space S_{2k+1} is determined by the $2k+2$ points $x_1, \ldots, x_{k+1}, x_1', \ldots, x_{k+1}'$. It has been tacitly assumed that these points are linearly independent, and that $2k+1 \gtreqless n$. If there should happen to be h independent linear relations among the $2k+2$ points, then we should speak of the tangent space S_{2k+1-h} along the generator S_k of the variety V_{k+1}.

The point y given by equation (43) on a generator S_k of a variety V_{k+1} will be said to lie also on a *consecutive* generator in case there exist parameters w_i such that

$$(48) \qquad y = \Sigma u_i x_i = \Sigma (u_i + w_i dt)(x_i + x_i' dt) \qquad (i=1, \ldots, k+1),$$

the generator *consecutive* to S_k being regarded as determined by the points $x_i + x_i' dt$. Such a point y is called a *singular point*, or *focus*, of the generator

S_k, and the locus of the foci on S_k is called *the singular space* of this generator. Equations (46), (48) imply

$$z+\Sigma wx=0 \,, \tag{49}$$

infinitesimals of higher order than the first being neglected. Therefore the point z corresponding to a singular point y lies on the generator S_k containing the point y, and *the tangent space S_{k+1} at a focal point of a variety V_{k+1} is indeterminate.* Moreover, if there are h independent foci on a generator S_k, the linear space tangent to the variety V_{k+1} along S_k is a space S_{2k+1-h} called *the singular tangent space* along the generator S_k of the variety V_{k+1}. If the parameters w_i are functions of t, a focal point y describes a curve when t varies, and the tangent at a point of this curve lies in the generator S_k through the point.

The foregoing considerations can be generalized in the following way. Let us consider a variety V_{k+m} which is the locus of ∞^m linear spaces $S_k(k<n-m)$. Such a variety V_{k+m} can be represented analytically as before, except that the coordinates of the points $x_1, \ldots, x_{k+1}$ are now functions of m independent variables $t^1, \ldots, t^m$. Any point y on the variety V_{k+m} is still given by an equation of the form (43), but the locus of each point x_i is now naturally called *a director variety V_m.*

The differential geometry of a variety V_{k+m}, in the neighborhood of a generator S_k, can be studied by considering a variety V_{k+1} within the variety V_{k+m} and through the generator S_k, so that the preceding results can be applied. Such a variety V_{k+1} can be defined by placing

$$t^j=t^j(\tau) \qquad (j=1, \ldots, m) \,, \tag{50}$$

where τ is an independent variable which takes the place of t in the application of the preceding results. The point z, corresponding to a point y on the generator S_k and defined by equation (46), is now given by

$$z=\Sigma\Sigma u_i(\partial x_i/\partial t^j)(dt^j/d\tau) \qquad (i=1, \ldots, k+1; j=1, \ldots, m) \,. \tag{51}$$

The tangent space S_{k+1} at the point y on the generator S_k of the variety V_{k+1} is determined, as before, by the points $x_1, \ldots, x_{k+1}, z$. *The tangent space S_{2k+1} along the generator S_k of the variety V_{k+1}* is easily shown to be determined by the points

$$\begin{cases} x_1, \ldots, \quad x_{k+1}\,, \quad \Sigma(\partial x_1/\partial t^j)(dt^j/d\tau)\,, \ldots, \\ \qquad \Sigma(\partial x_{k+1}/\partial t^j)(dt^j/d\tau) \qquad (j=1, \ldots, m) \,. \end{cases} \tag{52}$$

It may be noted here that fixing τ fixes the generator S_k; the values of u_i associated with this value of τ determine the point y on S_k; the form of the functions $t^j(\tau)$ determines the variety V_{k+1} within the variety V_{k+m}; and when the variety V_{k+1} varies in all possible ways through the fixed generator S_k, the derivatives $dt^j/d\tau$, for the fixed value of τ, vary.

The tangent space S_{k+m} at a point P_y of the variety V_{k+m} is defined in the usual way and is determined by the points

$$y, \quad \partial y/\partial u_i, \quad \partial y/\partial t^j \quad (i=1, \ldots, k+1; j=1, \ldots, m),$$

of which only $k+m+1$ are linearly independent. Indeed, this space S_{k+m} is found to be determined by the $k+m+1$ points

$$x_i, \quad \Sigma u_i(\partial x_i/\partial t^1), \ldots, \quad \Sigma u_i(\partial x_i/\partial t^m) \quad (i=1, \ldots, k+1). \tag{53}$$

Moreover, *the tangent space S_{k+m} could have been defined as the ambient of the tangent space S_{k+1} at the point P_y of the variety V_{k+1} as V_{k+1} varies in all possible ways through the generator S_k containing P_y.* The tangent space S_{k+m} projects from the generator S_k the space S_{m-1} of the points

$$\Sigma u_i(\partial x_i/\partial t^1), \ldots, \quad \Sigma u_i(\partial x_i/\partial t^m) \quad (i=1, \ldots, k+1),$$

which is the locus of the point z when the variety V_{k+1} varies.

The tangent space $S_{(k+1)(m+1)-1}$ along a generator S_k of the variety V_{k+m} is defined to be the ambient of the tangent space S_{2k+1} along the generator S_k of a variety V_{k+1} through S_k, as the variety V_{k+1} varies in all possible ways through the generator S_k and within the variety V_{k+m}. The tangent space $S_{(k+1)(m+1)-1}$ may also be characterized as the ambient of the tangent space S_{k+m} at a point y on the generator S_k of the variety V_{k+m} as the point y varies over the generator S_k, and is determined by the $(k+1)(m+1)$ points

$$x_i, \quad \partial x_i/\partial t^j \quad (i=1, \ldots, k+1; j=1, \ldots, m). \tag{54}$$

The dimensions of the tangent space along a generator S_k of a variety V_{k+m} would be less than $(k+1)(m+1)-1$ if there were one or more foci on each generator S_k.

56. The locus of the tangent planes of a surface. An interesting illustration of the theory in the latter part of the last section is found in the special case of the locus of the tangent planes of a surface immersed in space S_n $(n>4)$. Since a plane is a space S_2, and since there are ∞^2 points on a surface, we place $k=m=2$ in the theory of a variety V_{k+m} and observe at once that *the locus of the tangent planes of a surface is ordinarily a variety*

V_4. If the parametric vector equation of the surface is $x=x(u, v)$, then *the parametric vector equation of the variety* V_4 can be written in the form

$$y=x+sx_u+tx_v \,, \tag{55}$$

where s, t, u, v are independent variables. For the purpose of specializing the preceding discussion we place $u_1=1$, $u_2=s$, $u_3=t$; $x_1=x$, $x_2=x_u$, $x_3=x_v$; $t^1=u$, $t^2=v$.

Let us notice the special cases in which the locus of the tangent planes of a surface has dimensions less than four, and then exclude these cases from further consideration. Of course, if the surface were a plane, the locus would be simply this plane. Moreover, if the surface were a developable not a plane, or if it were a non-developable surface in space S_3, the locus of its tangent planes would be a variety V_3. The converse of the last statement is also true. For, if the locus is a variety V_3, the matrix of five columns and $n+1$ rows made of y, y_s, y_t, y_u, y_v is of rank four. Differentiation gives

$$y_s=x_u \,, \qquad y_t=x_v \,, \qquad y_u=x_u+sx_{uu}+tx_{uv} \,, \qquad y_v=x_v+sx_{uv}+tx_{vv} \,. \tag{56}$$

Hence the vanishing of a general one of the fifth order determinants of the matrix may be expressed by the equation

$$(x,\ x_u,\ x_v,\ sx_{uu}+tx_{uv},\ sx_{uv}+tx_{vv})=0 \,,$$

which is an identity in s, t, u, v. Using the fact that it is an identity in s, t, one sees that x satisfies two independent equations of the second order of the form (IV, 10). Therefore the surface is developable or else is a non-developable surface immersed in space S_3, as was to be proved.

The tangent space S_4 at a point y of the variety V_4 associated with a surface S is determined by the points y, y_s, y_t, y_u, y_v, now supposed to be linearly independent. Since this space S_4 depends not on s, t separately but only on the ratio s/t, it follows that *at every point on a tangent line at a point x of the surface S the tangent space S_4 of the variety V_4 is the same.* Therefore *the variety V_4 has at most ∞^3 tangent spaces S_4.*

It is easy to show that two *consecutive* tangent planes of a surface have at least one point in common. In fact, the two tangent planes may be regarded as determined by the two sets of three points

$$\begin{cases} x, \qquad x_u, \qquad x_v \,; \\ x+x_udu+x_vdv \,, \qquad x_u+x_{uu}du+x_{uv}dv \,, \qquad x_v+x_{uv}du+x_{vv}dv \,. \end{cases} \tag{57}$$

The point $x+x_u du+x_v dv$ is obviously in both planes. Consequently, *the contact point of a tangent plane is a focus of the plane regarded as a generator of the variety* V_4.

The tangent planes of a surface S constructed at the points of a curve

$$(58) \qquad u=u(\tau)\ , \qquad v=v(\tau)$$

on S form a variety V_3 within the variety V_4. The tangent space S_3 at a point y of this variety V_3 is determined by the points $x,\ x_u,\ x_v,\ z$, where z is given, according to equation (51), by

$$(59) \qquad z=x_u u'+x_v v'+s(x_{uu}u'+x_{uv}v')+t(x_{uv}u'+x_{vv}v')\ ,$$

accents indicating differentiation with respect to τ. Since each tangent plane of the surface S has a focus, the linear space tangent to the variety V_3 along a tangent plane of the surface S is a space S_4 instead of the space S_5 that might have been expected. This space S_4 is determined, according to (52), by the points

$$(60) \qquad x, \qquad x_u, \qquad x_v, \qquad x_{uu}u'+x_{uv}v', \qquad x_{uv}u'+x_{vv}v'\ .$$

The tangent linear space along a generator of the variety V_4, i.e., along a tangent plane of the surface S, is ordinarily a space S_5 instead of the space S_8 that might have been expected. This tangent space S_5 is determined, according to (54), by the points $x,\ x_u,\ x_v,\ x_{uu},\ x_{uv},\ x_{vv}$, and is identical with the space $S(2,0)$ at the point x of the surface S.

If the fundamental surface is a surface F, the tangent space S_4 at a point y of the variety V_4 is independent of the ratio s/t, as can easily be verified by use of equations (56). Conversely, *if the space S_4 is independent of s/t, the surface is a surface F*, since, on setting $s=0$, $t=1$ in equations (56), one sees that the space of the points $x,\ x_u,\ x_v,\ x_{uv},\ x_{vv}$ must be the same as the space of the points $x,\ x_u,\ x_v,\ x_{uu},\ x_{uv}$ corresponding to $s=1$, $t=0$; therefore the surface is a surface F. Consequently, *the tangent space S_4 at a point of the variety V_4 which is the locus of the tangent planes of a surface F is the same at all points of a tangent plane of F*, so that this variety V_4 has only ∞^2 tangent spaces S_4, instead of the usual ∞^3 (see Ex. 16). Moreover, the tangent space along a generator of the variety V_4 (ordinarily a space S_5) is, for a surface F, this same space S_4, and is the space $S(2,0)$ as in the case of an unspecialized surface.

57. Sets of varieties which are loci of linear spaces, with the generators in correspondence. The projective differential geometry of certain sets of varieties in space S_n, which are loci of linear spaces and which have their

generators in correspondence, can be studied effectively by means of systems of linear homogeneous differential equations of the first order. When each variety is the locus of ∞^1 linear spaces, the differential equations are ordinary; otherwise partial differential equations are used.

This section will be concerned for the most part with the projective differential geometry that can be studied by means of a system of $n+1$ ordinary linear homogeneous differential equations of the first order in $n+1$ dependent variables. In the latter part of the section a system of $m(n+1)$ linear homogeneous partial differential equations of the first order in $n+1$ dependent variables and m independent variables will be introduced.

In space S_n let us consider $n+1$ linearly independent points $x_1, \ldots, x_{n+1}$ with projective homogeneous coordinates

$$x_i^j \qquad (i, j=1, \ldots, n+1) , \tag{61}$$

and let us first suppose that these coordinates are single-valued analytic functions of one independent variable t. Then, as t varies, the locus of each point x_i is a curve C_i, and we thus obtain $n+1$ curves with their points in correspondence, corresponding points being those that correspond to the same value of the parameter t.

The square matrix of the coordinates x_i^j of the points x_i is of order and rank $n+1$. Therefore it is possible to determine* the coefficients of a system of differential equations of the form

$$x_i' = \sum_{j=1}^{n+1} a_{ij} x_j \qquad (i=1, \ldots, n+1) \tag{62}$$

so that

$$(x_1^j, \ldots, x_{n+1}^j) \qquad (j=1, \ldots, n+1)$$

will be $n+1$ sets of solutions. For example, if each of these sets is substituted in turn in the first equation of system (62) with its $n+1$ coefficients a_{1j} regarded as unknown, the resulting $n+1$ linear algebraic equations can be solved uniquely for these coefficients. Similarly, the coefficients of the other n equations can be determined.

The transformation of dependent variables,

$$x_i = \lambda_i \bar{x}_i \qquad (i=1, \ldots, n+1; \lambda_i \text{ scalar} \neq 0) , \tag{63}$$

and the transformation of independent variable,

$$u = u(t) \qquad (u' \neq 0) , \tag{64}$$

* Lane, 1928. 5, p. 786.

leave each of the curves C_i invariant. We reach thus the following conclusion.

The projective differential geometry of a set of $n+1$ *curves in space* S_n *with their points in correspondence can be studied by means of the invariants and covariants of system* (62) *under the total transformation* (63), (64).

With a different choice of the transformation of dependent variables the geometry of a different configuration can be studied by means of the corresponding invariants and covariants of system (62). For example, if n is odd and if the transformation

$$
(65) \qquad \left\{ \begin{array}{ll} x_i = \sum_{j=1}^{2} \lambda_{ij} \bar{x}_j & (i=1, 2) , \\ x_k = \sum_{j=3}^{4} \lambda_{kj} \bar{x}_j & (k=3, 4) , \\ \cdots \quad \cdots & \cdots \\ x_l = \sum_{j=n}^{n+1} \lambda_{lj} \bar{x}_j & (l=n, n+1) \end{array} \right.
$$

is used instead of the transformation (63), the configuration composed of $(n+1)/2$ ruled surfaces with their generators in correspondence can be studied. In general, *the number of configurations in space* S_n *whose projective differential geometry can be studied by means of the invariants and covariants of system* (62) *under linear transformations of the dependent variables and the transformation* (64) *of the independent variable is* $n(n^2+20)/24$ *if* n *is even, and is* $n(n^2+23)/24$ *if* n *is odd.* These formulas can be obtained by observing that the number under consideration is the number of ways in which it is possible to choose at least two transformations of the form

$$
(66) \qquad \left\{ \begin{array}{ll} x_i = \sum_{j=1}^{a} \lambda_{ij} \bar{x}_j & (i=1, \ldots, a) , \\ x_k = \sum_{j=a+1}^{b} \lambda_{kj} \bar{x}_j & (k=a+1, \ldots, b) , \\ \cdots \quad \cdots & \cdots \\ x_l = \sum_{j=g+1}^{h} \lambda_{lj} \bar{x}_j & (l=g+1, \ldots, h) , \end{array} \right.
$$

where $a, b, \ldots, g, h$ are positive integers such that

$$a \gtreqless b-a \gtreqless \ldots \gtreqless h-g \,, \qquad a<b<\ldots<g<h=n+1 \,.$$

Certain curves called *intersector curves* will now be defined and studied. Let us consider any one of the possible configurations just indicated in space S_n and any one of the varieties V_{k+1} in this configuration, which is the locus of a space S_k with $1 \leqq k \leqq n-2$. *A curve on such a variety V_{k+1} will be called an intersector curve with respect to the remaining varieties in the configuration in case the tangent to the curve at the point where it crosses each generator S_k intersects the space S_{n-k-1} determined by the corresponding generators of the other varieties.* In order to obtain the differential equation of these curves let us observe that any curve on the variety V_{k+1}, with certain non-essential exceptions, can be regarded as the locus of the point P_y defined by

$$y=x_1+\sum_{j=2}^{k+1} u_j x_j \,, \tag{67}$$

in which the non-homogeneous parameters $u_2, \ldots, u_{k+1}$ are functions of the independent variable t. The point $y'+hy$ is any point (except P_y) on the tangent of this curve at P_y; by means of equations (62), $y'+hy$ can be expressed as a linear combination of $x_1, \ldots, x_{n+1}$. The curve is an intersector curve if the coefficients of $x_1, \ldots, x_{k+1}$ in this expression vanish. Thus one obtains $k+1$ equations, and the elimination of h therefrom results in *the differential equations of the intersector curves*,

$$u'_p+a_{1p}-a_{11}u_p+\sum_{j=2}^{k+1} u_j(a_{jp}-u_p a_{j1})=0 \qquad (p=2, \ldots, k+1) \,. \tag{68}$$

There are ∞^k of these curves, one of them passing through each point of each generator S_k.

The tangents of the intersector curves may be called *intersector tangents*. The locus of the intersector tangents at the points of a fixed generator S_k is a variety V_{k+1} whose local equations can be found in the following way. In the expression for $y'+hy$ as a linear combination of $x_1, \ldots, x_{n+1}$ let us replace u'_p by the expression given therefor by equation (68). Then let us denote the coefficients of $x_1, \ldots, x_{n+1}$ by $y_1, \ldots, y_{n+1}$ respectively, so that $y_1, \ldots, y_{n+1}$ are local coordinates of the point $y'+hy$ on an intersector tangent referred to the pyramid whose vertices are the points

$x_1, \ldots, x_{n+1}$. Thus we find *the parametric equations of the locus of the intersector tangents at the points of a fixed generator* S_k,

$$(69) \qquad \begin{cases} y_1 = a_{11} + \sum_{j=2}^{k+1} u_j a_{j1} + h \;, \\ y_p = u_p y_1 & (p = 2, \ldots, k+1) \;, \\ y_q = a_{1q} + \sum_{j=2}^{k+1} u_j a_{jq} & (q = k+2, \ldots, n+1) \;. \end{cases}$$

If $2k > n-2$, homogeneous elimination of $u_2, \ldots, u_{k+1}$ and h from these equations gives $n-1-k$ quadratic equation of the locus of the intersector tangents,

$$(70) \qquad \sum_{j=1}^{k+1} y_j (a_{jq} y_{n+1} - a_{j,\,n+1} y_q) = 0 \qquad (q = k+2, \ldots, n) \;.$$

If, however, $2k \lesseqgtr n-2$, homogeneous elimination of u_j from the last group of equations (69) gives $n-1-2k$ linear equations; then k quadratic equations can be obtained as before.

We turn now to the consideration of a special case. The configuration composed of two ruled surfaces with their generators in one-to-one correspondence, and with corresponding generators skew to each other, is of frequent occurrence in the geometry of surfaces in ordinary space. Such a configuration can be studied by means of the equations (62) when $n = 3$ and the transformation (65) is used. The differential equations can be reduced by this transformation to the canonical* form illustrated by figure 41,

$$(71) \qquad \begin{cases} x_1' = a_{13} x_3 + a_{14} x_4 \;, & x_3' = a_{31} x_1 + a_{32} x_2 \;, \\ x_2' = a_{23} x_3 + a_{24} x_4 \;, & x_4' = a_{41} x_1 + a_{42} x_2 \;. \end{cases}$$

Let us denote the two surfaces by R_{12} and R_{34}, their generators being the lines $x_1 x_2$ and $x_3 x_4$. The intersector curves on the ruled surface R_{12} with respect to the ruled surface R_{34} are those curves such that the tangent line at each point $x_1 + u x_2$, on a generator $x_1 x_2$, of each of them intersects the corresponding generator $x_3 x_4$. The differential equation of the intersector curves on the surface R_{12} is found to be an equation of Riccati,

$$(72) \qquad u' = -a_{12} + (a_{11} - a_{22}) u + a_{21} u^2 \;.$$

* Lane, 1923. 4, p. 284.

This differential equation becomes $u'=0$ when the canonical form (71) is being used; hence the curves $u=\text{const.}$ are intersector curves on the surface R_{12}; a similar remark can be made for the surface R_{34}. The locus of the

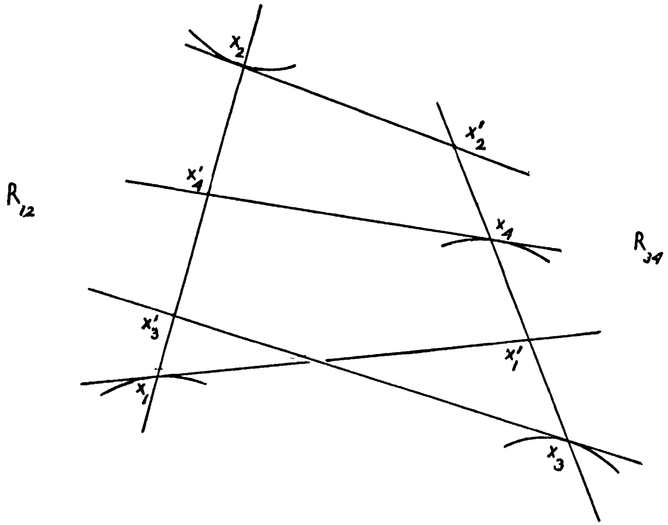

FIG. 41

intersector tangents of the surface R_{12} at the points on a generator x_1x_2 has the local equation

$$a_{14}y_1y_3-a_{13}y_1y_4+a_{24}y_2y_3-a_{23}y_2y_4=0 \tag{73}$$

and is therefore a quadric surface.

The foregoing discussion can be generalized in the following way. If in (61) the coordinates x_i^j of the $n+1$ points x_i are now supposed to be functions of m (and not fewer than m) independent variables $t^1, \ldots, t^m$ $(m<n)$, the locus of each of the points is a variety V_m, and we thus obtain $n+1$ such varieties with their points in correspondence. It is possible to determine the coefficients of a system of partial differential equations of the form

$$\partial x_i/\partial t^p=\sum_{j=1}^{n+1} a_{ij\,p}x_j \qquad (i=1, \ldots, n+1;\, p=1, \ldots, m) \tag{74}$$

so that

$$(x_1^j, \ldots, x_{n+1}^j) \qquad\qquad (j=1, \ldots, n+1)$$

will be $n+1$ sets of solutions. These coefficients satisfy $m(m-1)(n+1)^2/2$ integrability conditions which can be obtained by demanding that the expressions for the derivatives of the second order calculated by differentiating equations (74) shall be unique. The *integrability conditions* are

$$\tag{75} \begin{cases} \partial a_{ijp}/\partial t^q + \sum_{r=1}^{n+1} a_{irp}a_{rjq} = \partial a_{ijq}/\partial t^p + \sum_{r=1}^{n+1} a_{irq}a_{rjp} \\ \qquad (i, j=1, \ldots, n+1;\ p, q=1, \ldots, m;\ p<q) , \end{cases}$$

and from them it is easy to deduce the equations

$$\tag{76} \sum_{i=1}^{n+1} (\partial a_{iip}/\partial t^q - \partial a_{iiq}/\partial t^p) = 0 \qquad (p, q=1, \ldots, m;\ p<q) .$$

By means of a transformation of the form (66) with coefficients that are functions of $t^1, \ldots, t^m$, and a transformation of independent variables of the form

$$\tag{77} u^p = u^p(t^1, \ldots, t^m) \qquad (p=1, \ldots, m) ,$$

we can study the projective differential geometry of a configuration which consists of a certain number of varieties each of which is the locus of ∞^m linear spaces in correspondence.

EXERCISES

1. The space S_5 of the points (10) is the limit of the space S_5 determined by the tangent plane at a point P_x of a surface and the tangent plane at a neighboring point Q on a curve C through P_x, as Q approaches P_x along C.

2. Consider the space S_6 which joins the space $S(2, 0)$ at a point P_x of a surface S to a variable point H of the cubic curve in the space S_3 of the points $x_{uuu}, \ldots, x_{vvv}$, and prove that when H varies on the cubic, the space S_6 generates a cone which is a variety V_7^3.

3. Discuss in detail the neighborhood of the fourth order of a point on a surface in space S_n.

4. The space $S(2, 1)$ at a point P_x of a variety V_m in the direction of a curve through P_x on V_m is a space S_{m+1}. The variety $V(2, 0)$ which is the generalized cone of Del Pezzo at the point P_x of the variety V_m has for its vertex the tangent space S_m of V_m at P_x, lies in the space $S(2, 0)$ of V_m at P_x, and is of order 2^{m-1}.

5. A hyperplane of tacnodal section at a point P_x of a surface in space S_5 is tangent to the cone of Del Pezzo at P_x, and is also tangent to the cone of Del Pezzo at a point consecutive to the point P_x on a principal curve through P_x.

SEGRE, 1921. 1, p. 201

6. At a point of a surface the triple of directions defined by a non-singular binary cubic differential form $adu^3+bdv^3=0$ has for hessian pair the pair of directions given by $dudv=0$. The directions of the hessian pair separate harmonically the directions of the first polar pair $a\delta udu^2+b\delta vdv^2=0$ of an arbitrary direction $\delta v/\delta u$ with respect to the triple. This property is characteristic of the hessian pair.

7. When the hyperplane $S(2, 0)$ at a point consecutive to a point P_x on a curve in the direction dv/du at P_x, on a surface F in space S_5, intersects the tangent plane of F at P_x in the tangent line of a curve in the direction $\delta v/\delta u$ through P_x, then the direction dv/du is one or the other of the polar pair (31) of the direction $\delta v/\delta u$ with respect to the triple of directions (24).

8. Construct a theory of surfaces in space S_4 using a system of equations of the form

$$x_{uv}=cx+ax_u+bx_v\ ,$$

$$x_{uuu}=px+\alpha x_u+\beta x_v+\rho x_{uu}+\sigma x_{vv}\ ,$$

$$x_{vvv}=qx+\gamma x_u+\delta x_v+rx_{uu}+sx_{vv}\ .$$

Find the conditions of complete integrability, and compute power series expansions for a point on the surface referred to the local pyramid of reference with vertices at the points x, x_u, x_v, x_{uu}, x_{vv}. Considering the effect of the transformation

$$x=\lambda\bar{x}\ ,\qquad \bar{u}=U(u)\ ,\qquad \bar{v}=V(v)\ ,$$

reduce the system of equations to canonical forms; find a covariant pyramid of reference, and a canonical form for the power series expansions.

9. On a surface of Exercise 8, find the differential equation of the ∞^2 quasi-asymptotic curves each of which is such that at each of its points its osculating space S_3 coincides with the space $S(2, 1)$ in the direction of the curve at the point. Prove that every space S_3 containing the tangent plane at a point of the surface is the osculating space S_3 of two of these quasi-asymptotic curves through the point, whose directions separate the conjugate directions at the point harmonically.

BOMPIANI, 1912. 2, p. 404

10. Considering those surfaces of space S_4 which sustain one-parameter families of asymptotic curves, interchange x_{uv} and x_{uu} in the fundamental equations of Exercise 8 and reconstruct the theory.

11. Two *consecutive* spaces $S(k, 0)$ intersect in a space $S(k-1, 0)$ and are contained in a space $S(k+1, 0)$.

12. Extend the correlation of Chasles to read as follows. The tangent spaces S_{k+1} at the points on a generator S_k of a variety V_{k+1} correspond projectively to their points of contact.

13. Discuss the variety generated by the point z given in equation (51) when u_i and $dt^j/d\tau$ vary and τ is fixed. This variety is ordinarily of dimensions $k+m-1$ and of order $C_{k+m-1,k}$; it is of class $C_{k+1,m-1}$ or $C_{m,k}$ according as $m \overline{\underline{<}} k+1$ or $m \overline{\underline{>}} k+1$.
SEGRE 1910. 1, p. 93

14. On a surface $x=x(u, v)$ in space S_5 consider three *consecutive* points P_x, P_1, P_2 of a curve belonging to a one-parameter family of curves. The tangent planes of the surface at the points P_1, P_2 determine a space S_4 which intersects the tangent plane of the surface at the point P_x in a straight line. Denoting the direction of this line by $\delta v/\delta u$, show that

$$(x,\ x_u,\ x_v,\ x_{uu}du+x_{uv}dv,\ x_{uv}du+x_{vv}dv\ ,$$

$$\delta u[x_{uuu}du^2+2x_{uuv}dudv+x_{uvv}dv^2+x_{uu}d^2u+x_{uv}d^2v]$$
$$+\delta v[x_{uuv}du^2+2x_{uvv}dudv+x_{vvv}dv^2+x_{uv}d^2u+x_{vv}d^2v])=0\ .$$

(The ∞^1 given curves and the ∞^1 curves defined by this equation are said to be in the relation of *conjugacy of the second kind.*) Those curves on the surface which are self-conjugate in this sense are the principal curves. BOMPIANI, 1922. 6

15. Using the usual local coordinate system in the space $S(2, 0)$ at a point x of a surface S, show that the equation of the space S_4 which is tangent, at the point y of equation (55), to the variety V_4 which is the locus of the tangent planes of S is

$$t^2x_4+stx_5+s^2x_6=0\ .$$

Hence show that the one-parameter family of these spaces S_4 at all points of the tangent plane at the point x of the surface S envelop the cone of Del Pezzo, $x_5^2-4x_4x_6=0$.

16. If a surface immersed in space S_5 is non-developable and is not a surface F, and if the variety V_4 which is the locus of its tangent planes admits only ∞^2 tangent spaces S_4, then the surface is a surface of Veronese. SEGRE, 1907. 2, p. 1078

17. Consider a curve C on a non-developable surface S, which is not a surface F and is immersed in space S_5. Consider the variety V_3 which is the locus of the tangent planes of S at the points of C. Consider, further, the space S_4 which is tangent to the variety V_3 along the tangent plane at a point x of the surface S. This space S_4 has four-point contact with the curve C at the point x and consequently contains the osculating space S_3 at the point x of the curve C if, and only if, the curve C is a principal curve. SEGRE, 1921. 1

18. A plane curve on a surface immersed in space S_5 is a principal curve. If a non-developable surface immersed in space S_5 has on it ∞^2 plane curves, the principal curves are indeterminate, and the plane curves are conics.

19. Calculate the integrability conditions of system (33), and prove that if the principal curves are indeterminate on a non-developable surface immersed in space S_5, the surface is a surface of Veronese.

20. At a point x of a surface $x=x(u, v)$ in space S_n consider the parametric curves C_u, C_v. The ruled surface R_u of u-tangents constructed at the points of the curve C_v has for tangent space S_3 along the generator xx_u the space S_3 determined by the points x, x_u, x_v, x_{uv}. The ruled surface R_v of v-tangents constructed at the points of the curve C_u has along the generator xx_v the same tangent space S_3.

21. There are two lines that are tangent to both of two ruled surfaces R_{12}, R_{34} with their generators in one-to-one correspondence in space S_3, the two points of contact with R_{12} being on any prescribed generator x_1x_2, and the two points of contact with R_{34} being on the corresponding generator x_3x_4. Determine these four points of contact. LANE, 1923. 4, p. 291

22. When a point x varies on a curve of the family $dv-\lambda du=0$ on an integral surface of equations (III, 6), the line joining the point x to the point y defined by equation (III, 35) generates a ruled surface R_{xy}, and the reciprocal line joining the points ρ, σ defined by equations (III, 42) describes a ruled surface $R_{\rho\sigma}$. These surfaces have their generators in one-to-one correspondence, and the sixteen coefficients of the system of equations of the form (62) for these surfaces have the following expressions when $x_1=x$, $x_2=y$, $x_3=\rho$, $x_4=\sigma$:

$$
\begin{aligned}
&a_{11}=b+a\lambda\,, \qquad a_{12}=0\,, \qquad a_{13}=1\,, \qquad a_{14}=\lambda\,,\\
&a_{21}=p_v-ap+q\beta+bA+a(F-2a\beta+\beta\psi)+[q_u-bq+p\gamma+b(G-2b\gamma+\gamma\varphi)+aB]\lambda\,,\\
&a_{22}=\theta_u-b+(\theta_v-a)\lambda\,, \qquad a_{23}=A+(G-2b\gamma+\gamma\varphi)\lambda\,,\\
&a_{24}=G-2a\beta+\beta\psi+\lambda B\,, \qquad a_{31}=F+(ab-b_v)\lambda\,,\\
&a_{32}=\lambda\,, \qquad a_{33}=\theta_u-b+a\lambda\,, \qquad a_{34}=\beta\,,\\
&a_{41}=ab-a_u+\lambda G\,, \qquad a_{42}=1\,, \qquad a_{43}=\gamma\lambda\,, \qquad a_{44}=b+(\theta_v-a)\lambda\,.
\end{aligned}
$$

23. Using equations (62) and the transformation (65) with $n=5$, study a triple of ruled surfaces R_{12}, R_{34}, R_{56} in space S_5 with their generators x_1x_2, x_3x_4, x_5x_6 in correspondence. In particular, show that on the ruled surface R_{12} there are two curves such that at each point y, defined by $y=x_1+ux_2$ on a generator x_1x_2, of each of them its osculating plane meets in a straight line the space S_3 of the corresponding generators x_3x_4, x_5x_6. These curves are generated by the points y for which u is a root of the equation

$$\sum_{j=3}^{6} a_{2j}a_{j1}u^2+(a_{1j}a_{j1}-a_{2j}a_{j2})u-a_{1j}a_{j2}]=0\,.$$

LANE, 1928. 5, p. 790

24. At the conclusion of Section 57 let $n=3$, $m=2$, and develop a theory of pairs of congruences in ordinary space with their generators in one-to-one correspondence.

Calculate the thirty-two coefficients of the system of equations of the form (74) for a pair of reciprocal congruences associated with a surface in ordinary space (see Ex. 22).
COOK, 1930. 2

25. Consider an integral surface of the system of equations

$$x_{uuv} = ax_{uu} + hx_{uv} + bx_{vv} + lx_u + mx_v + dx ,$$
$$x_{uvv} = a'x_{uu} + h'x_{uv} + b'x_{vv} + l'x_u + m'x_v + d'x ,$$

and suppose that this surface is not an integral surface of any equation of the second order. Two of the eight integrability conditions of this system are $a' = b = 0$. Consider, at a point of the integral surface, the two ruled surfaces R_u, R_v described in Exercise 20. Consider, on the surface R_u, all the curves that intersect the generator xx_u, and prove that the osculating planes of these curves at the points of the generator xx_u lie in a space S_4. Prove the symmetric theorem with u and v interchanged.
BOMPIANI, 1919. 2, pp. 362–63

26. The ∞^2 spaces S_3 defined in Exercise 20 at the points of an integral surface of the system of Exercise 25 can be arranged in two ways, along the u-curves and along the v-curves, as the ∞^1 osculating spaces S_3 of ∞^1 curves.
BOMPIANI, 1919. 2, p. 633

27. As a point x varies along a v-curve on an integral surface S of the system of Exercise 25, the point $x_u - b'x$ describes a curve characterized geometrically by the property that its osculating plane at each of its points is in the space S_3 tangent, along the line xx_u through the point, to the ruled surface R_u. As u, v vary, the point $x_u - b'x$ describes a surface of the same type as the surface S. The point $x_v - ax$ describes similarly a transform of the surface S in the v-direction.
BOMPIANI, 1919. 2, p. 634; P. TZITZÉICA, 1927. 2

28. In the space S_6 of the points $x_{uu}, \ldots, x_{vvv}$ of Section 52 the parametric equations of the variety V_3, which is the locus of the plane determined by the points $x_{uu} + x_{uv}\lambda$, $x_{uv} + x_{vv}\lambda$, H, can be written in the form

$$y_4 = h , \quad y_5 = h\lambda + k , \quad y_6 = k\lambda ,$$
$$y_7 = l , \quad y_8 = 3l\lambda , \quad y_9 = 3l\lambda^2 , \quad y_{10} = l\lambda^3 .$$

Writing three linear equations which represent a fixed space S_3 in the space S_6, prove analytically that the variety V_3 meets the space S_3 in five points, thus proving again that the variety V_3 is of order 5.

29. The space $S(2, 0)$ at a point P_x of a surface F is filled by the osculating planes at P_x of all the curves on F that pass through P_x.

CHAPTER VIII

MISCELLANEOUS TOPICS

Introduction. The purpose of this chapter is to include certain topics which it does not seem desirable to omit altogether, and which nevertheless do not find adequate presentation elsewhere in this volume. Section 58 is devoted to some historical remarks sketching the beginning and growth of projective differential geometry. The method of differential forms is discussed in Section 59 more extensively than hitherto in these pages, particular attention being given to the way in which the method has been applied by Fubini in the study of surfaces in ordinary space. In Section 60 certain coordinate systems are compared and the equations of transformation between them are found. Finally, in Section 61 a brief account is given of Wilczynski's theory of congruences in ordinary space, a subject so intimately connected with the theory of surfaces that it seems appropriate to include it here.

58. Historical remarks. The purpose of this section is to recapitulate briefly the story of the beginning and growth of projective differential geometry. There was first of all a period of discovery of isolated theorems which were of a projective differential nature but which were not recognized at the time as having this character, because there was then no organized science of projective differential geometry. Later came the initiation of projective differential investigations, and still later the organization of comprehensive theories and the perfection of systematic methods of study.

Perhaps the simplest and earliest example of a configuration whose definition is of a projective differential nature consists of the tangent line at a point of a curve. The definition that we use today does not go back to the Greeks, who seem to have thought of the tangent at a point of a curve as merely cutting the curve in one point at its point of contact. Our definition of the tangent as the limit of a secant probably dates no farther back than the works of Fermat (1601–65) and Descartes (1596–1650).

Since a developable surface is the locus of the tangents of a curve it is natural, and quite probable, that the theory of developable surfaces bears the next earliest date. Cajori* says: "The first critical studies of developable surfaces were made by Leonhard Euler and Gaspard Monge. The two investigators approached the subject about the same time, but Euler's

* Cajori, 1929. 3, p. 432.

paper* received earlier publication, in 1772. It is noteworthy that at this time Euler was blind. . . . . About the same time, and independently of Euler, the subject of developable surfaces was investigated by Gaspard Monge, the creator of descriptive geometry. His earliest publication on such surfaces appeared at Paris in 1785; he discussed them repeatedly in later writings. Monge's treatment is less analytical than that of Euler and more nearly the result of direct contemplation of space relations." We may remark that Euler (1707–83) seems to have been the first to consider a developable surface as the locus of a one-parameter family of lines, and that Monge (1746–1818) seems to have been the first to consider a developable as the envelope of a one-parameter family of planes.

Monge made still other contributions to projective differential geometry. For example, he gave† the first propositions on general congruences of straight lines. In particular, he discovered the two focal points on each generator of a congruence, and the two families of developables in a congruence. Moreover, both in the memoir last cited and in his *Géométrie descriptive* (1798) he studied the lines of curvature on a surface which, to be sure, are not capable of a purely projective definition, but which are closely related to certain aspects of the projective theory. Monge is said to have been an inspiring teacher, and it is certain that some of his pupils distinguished themselves as geometers, among them being Malus, Dupin, Brianchon, Poncelet, and Plücker.

It seems appropriate to make a few remarks here on the chronology of the geometry that has the straight line as generating element. Malus in his *Optique* seems to have originated‡ the concept of a general complex of straight lines, but the name is due to Plücker. Giorgini was probably the first§ to consider, in 1827, the linear complex, which was studied in 1833 by Möbius and in 1837 by Chasles. Plücker in his *System der Geometrie des Raums* (1846) was the first to take the straight line as the generating element of ordinary space, but it was Grassmann who, in his *Ausdehnungslehre* (1844), defined the projective homogeneous coordinates of a linear subspace S_k of a linear space S_n $(k<n)$ by means of the coordinates of $k+1$ independent points in the space S_k. In the special case $n=3$, $k=1$ we have the line coordinates commonly called plückerian although Grassmann seems to have defined them first. Plücker used quite a different system of line coordinates in his early investigations of ordinary ruled space. It is worthy

* Euler, 1772. 1.

† Monge, 1781. 1.

‡ *Journal de l'école polytechnique*, Vol. VII (1808).

§ *Memorie di matematica della società italiana delle scienze* (Modena).

of note that Cayley in 1859 independently discovered Grassmann's coordinates of a straight line.

We have already had occasion to refer to various isolated results of a projective differential nature. We recall the work of Dupin on conjugate tangents and asymptotic tangents, bearing the date 1813. The correlation of Chasles dates back to 1839, and the theorem of P. Serret on ruled surfaces to 1860. Moutard discovered the quadric that bears his name in, or about, 1863. Hermite's theorem on the contact of a quadric surface with an analytic surface was published in 1873. Lie announced the discovery of the quadric that bears his name in 1878. And in 1880 Darboux described the tangents now called the tangents of Darboux.

In the presence of all these outcroppings of projective differential geometry it is not surprising to learn that at about the beginning of the last quarter of the nineteenth century the first attempt was made to construct a general theory of this subject. Credit for consciously undertaking the first systematic projective differential investigation is due to Halphen (1844–89). Reference has already been made to his Paris thesis of 1878 on plane curves, and to the memoir of 1880 on curves in ordinary space. These publications contain very fundamental and far-reaching results; surprisingly little has been added to our knowledge of plane and space curves since the time of Halphen.

The distinguished Italian geometer C. Segre (1863–1924) began his geometrical researches at the University of Torino in the early eighties of the nineteenth century. His interest in projective differential geometry is said to have been stimulated by Wilczynski at the Heidelberg Congress of 1904. Beginning with a very significant memoir in 1907 Segre made important contributions to the subject. He was not only interested in the geometry of ordinary space, to which his contributions of the *tangents of Segre* and the *cone of Segre* have already been studied in this book, but was a leader in studying the projective differential geometry of hyperspace. Segre gave analytic proofs regularly, but was also an outstanding exponent of the synthetic method, making differential properties even in hyperspace appear intuitive. This method has been used with great skill and success by Bompiani.

Wilczynski (1876——) turned his attention to geometry about 1901 and by 1906 had established his reputation as a geometer on a firm foundation. He contributed a systematic analytic method, namely, *the method of differential equations*, which has been used in a somewhat modified form in the greater part of this book. His extensive geometrical contributions have already been so thoroughly discussed in these pages that there is no need to enlarge on them here. One of the most successful exponents of Wilczynski's method was G. M. Green (1891–1919).

Fubini (1879——) became interested in projective differential geometry about 1914. He undertook to define a surface in ordinary space, except for a projective transformation, by means of differential forms. By 1916 he had perfected an analytical method in projective differential geometry, namely, *the method of differential forms*. His name is associated with that of one of the distinguished followers of his method, Edouard Čech, through their collaboration in publishing the well-known treatise.* Fubini's method is employed in their treatise, and we shall consider it again in the next section.

59. The method of Fubini. In the introduction to Chapter III we made some general comments on Fubini's method, in connection with the theory of surfaces in ordinary space. The purpose of this section is to follow up those remarks with a somewhat more detailed and explicit exposition of Fubini's method as applied to surfaces in ordinary space, and to make some further explanations of the power and limitations of the method.

We shall assume that the reader knows a little of the theory of differential forms. The absolute calculus of Ricci is very useful in this theory. In particular, the process of covariant differentiation with respect to a fundamental quadratic differential form is frequently employed. The formulas are very much simplified by use of the summation convention of tensor analysis, summation being understood with respect to any index that appears twice in the same term.

We propose, with Fubini, to define a surface in ordinary space by means of differential forms. Let us consider a fundamental binary quadratic differential form G,

$$G=a_{ij}du^i du^j \qquad (a_{ij}=a_{ji};\ i,\ j=1,\ 2)\ . \tag{1}$$

The coefficients a_{ij} are functions of the two variables u^1, u^2 in a certain region, and the discriminant A of the form is supposed not to vanish in this region. The form G will be more completely specified later on. Let us also consider in space S_3 a surface S whose parametric vector equation in projective homogeneous coordinates is

$$x=x(u^1,\ u^2)\ . \tag{2}$$

Let us now define two differential forms F_2, Φ_3 by the equations

$$F_2=(x,\ x_1,\ x_2,\ d^2x)\,|A\,|^{-1/2}\ , \qquad \Phi_3=(x,\ x_1,\ x_2,\ d^3x)\,|A\,|^{-1/2}\ , \tag{3}$$

wherein numerical subscripts of x denote covariant differentiation with respect to the form G, parantheses indicate determinants of the fourth order

* Fubini and Čech, 1926. 1 and 1927. 1.

of which only a typical row is written in each case, and vertical bars indicate absolute value. The forms F_2, Φ_3 are of the first and second orders respectively, since F_2 is independent of the second differentials of u^1, u^2, and Φ_3 is independent of the third differentials.

Both forms F_2, Φ_3 can be shown by direct calculations, which we shall omit, to be absolutely invariant under all proper transformations of parameters,

$$u^1 = u^1(\bar{u}^1, \bar{u}^2) , \qquad u^2 = u^2(\bar{u}^1, \bar{u}^2) . \tag{4}$$

To express this invariantive property the forms F_2, Φ_3 are called *intrinsic.*

One undesirable feature of the present situation is that the forms F_2, Φ_3 are of different orders. It is possible to replace Φ_3 by an intrinsic form of the first order. In fact, a little calculation, which again we shall omit, suffices to show that the form f_3 defined by

$$f_3 = 2\Phi_3 - 3dF_2 \tag{5}$$

is actually of the first order, being independent of the second differentials of u^1, u^2. We now have two binary differential forms F_2, f_3 one quadratic and the other cubic. These forms are independent of the parametric representation of the surface S, but still depend on the form G and on the proportionality factor λ of the homogeneous coordinates x.

We wish to find precisely how the forms F_2, f_3, and incidentally Φ_3, depend on the form G and the factor λ. Replacing G by another form G' and indicating the new expressions by accents, we find

$$F_2' = RF_2 , \qquad \Phi_3' = R\Phi_3 , \qquad f_3' = Rf_3 - 3F_2 dR , \tag{6}$$

where

$$R = |A/A'|^{1/2} . \tag{7}$$

Moreover, multiplying each coordinate x by a factor λ, i.e., making the transformation

$$\bar{x} = \lambda x , \tag{8}$$

and distinguishing the new expressions by dashes, we find

$$\bar{F}_2 = \lambda^4 F_2 , \qquad \bar{\Phi}_3 = \lambda^4 \Phi_3 + 3F_2\lambda^3 d\lambda , \qquad \bar{f}_3 = \lambda^4 f_3 - 6F_2\lambda^3 d\lambda . \tag{9}$$

Since the form f_3 is not transformed in the same way as the form F_2 under these transformations we inquire whether it is possible to replace f_3 by

another form having all the desirable properties possessed by f_3 and having also the property of being cogredient to F_2 under these transformations. The answer is in the affirmative. Let us denote the discriminant of F_2 by D, and define a form F_3 by the equation

$$F_3 = 2\Phi_3 - 3dF_2 + (3/4)F_2 d\log(D/A) \,. \tag{10}$$

Now it is easy to verify that under the transformations and with the notations of the preceding paragraph we have

$$F'_i = RF_i\,, \qquad \bar{F}_i = \lambda^4 F_i \qquad (i=2,\,3)\,. \tag{11}$$

Thus we reach the conclusion:

The forms F_2, F_3 are of the first order and of degrees indicated by the subscripts, are intrinsic, and are cogredient and relatively invariant under change of fundamental form G and the transformation (8) *of proportionality factor.*

We next choose the proportionality factor λ in a special way. Precisely, we choose λ so that the discriminant of the form $\bar{F}_3$ shall be a non-zero constant times the cube of the discriminant of $\bar{F}_2$. That it is possible to make this choice is evident when one considers that the discriminant of F_3 is of the fourth degree in the coefficients of F_3 while the discriminant of F_2 is quadratic in the coefficients of F_2. Denoting these discriminants by Δ, D respectively we find

$$\bar{\Delta} = \lambda^{16}\Delta\,, \qquad \bar{D} = \lambda^8 D\,. \tag{12}$$

Then in order to have $\bar{\Delta} = \text{const.}\ \bar{D}^3$ it is sufficient to choose λ so that

$$\Delta = \text{const.}\ \lambda^8 D^3\,. \tag{13}$$

We are supposing $D \neq 0$. It follows that $\Delta \neq 0$, and in particular that F_3 is not the cube of a linear factor. *We denote by φ_2, φ_3 the two forms $\bar{F}_2$, $\bar{F}_3$ given by the second of equations* (11) *with λ satisfying* (13).

The fundamental form G has hitherto been arbitrary. *We now choose G to be the form φ_2.* With this choice of G, direct calculation, which we shall omit, shows that the hessian of the form φ_3 is proportional to φ_2, so that φ_3 *is apolar to* φ_2. The coordinates x as now normalized are *Fubini's normal coordinates.* The forms φ_2, φ_3 are not only intrinsic but are covariant to the surface.

We can simplify the analysis by taking the asymptotic curves on the surface S as parametric. The reader may have observed already that *the curvilinear differential equation of the asymptotic curves is $\varphi_2 = 0$.* Let us write

$$\varphi_2 = 2\beta\gamma\, du\, dv\,. \tag{14}$$

It follows that we must have, except possibly for a constant factor,

$$\varphi_3 = 2\beta\gamma(\beta du^3 + \gamma dv^3) \; . \tag{15}$$

Therefore *the curvilinear differential equation of the curves of Darboux is* $\varphi_3 = 0$.

The question now is whether these two forms are sufficient to determine the surface S except for a projective transformation. The answer is in the negative. It is not difficult to show that the coordinates x satisfy equations of the form (III, 6), but that the surface is now determined only except for a projective applicability, the coefficients p, q of (III, 6) being not yet determined. *If we adjoin to the forms* φ_2, φ_3 *the form* ψ_2 *defined by*

$$\psi_2 = p du^2 - q dv^2 \; , \tag{16}$$

these three forms suffice to determine the surface S *except for a projective transformation,* since now all the coefficients of equations (III, 6) are determined. The form ψ_2 vanishes for the curves corresponding to the developables of the congruence which is reciprocal to the projective normal congruence of the surface S.

Fubini's method, which we have just explained for surfaces in ordinary space, has also been used by Sannia in studying plane and space curves; by Čech in his theory of ruled surfaces; and by Fubini in studying congruences and complexes in ordinary space, surfaces in space S_4, and hypersurfaces in space S_n. This method is available for any configuration that has a quadratic form covariantly connected with it. The method has failed for a variety V_k in space S_n with $1<k<n-1$ and $n>4$, either because of the lack of a covariant quadratic form, or because of the lack of an absolute calculus for an n-ary p-adic differential form. The method of Wilczynski is theoretically available even in these cases but the labor involved in applying this method seems to be largely prohibitive.

60. Comparison of certain coordinate systems. Aside from minor variations in notations and conventions which it is easy to recognize, there are two or three different local coordinate systems which have been used in studying surfaces in ordinary space and which it may be helpful to discuss here. We shall first of all compare the local coordinate system used in Chapter III with a local coordinate system much used by Fubini and Bompiani. Then we shall show how to pass from the notation and local coordinate system habitually employed by Wilczynski to that of Chapter III. Finally, we shall show how to transform the equations of Wilczynski's canonical quadric and cubic from his canonical coordinate system to the system used in Chapter III, and shall comment on the results.

Early in Chapter III we chose as vertices of a local tetrahedron of reference at a point x of an integral surface of equations (III, 6) the four covariant points x, x_u, x_v, x_{uv}, and chose the unit point so that a point X defined by

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv} \tag{17}$$

should have local coordinates proportional to $x_1, \ldots, x_4$. Fubini and Bompiani frequently use* as local coordinates of the same point X the four determinants Ω, N_2, N_1, T defined by

$$\begin{cases} \Omega = (X,\, x_u,\, x_v,\, x_{uv})\ , \qquad N_2 = (X,\, x,\, x_v,\, x_{uv})\ , \\ \qquad\qquad\qquad\qquad\qquad N_1 = (X,\, x,\, x_u,\, x_{uv})\ , \qquad T = (X,\, x,\, x_u,\, x_v)\ . \end{cases} \tag{18}$$

If the expression for X in equation (17) is substituted in equations (18) it is easy to see that *the equations of transformation between these two systems of coordinates are*

$$\rho x_1 = \Omega\ , \qquad \rho x_2 = -N_2\ , \qquad \rho x_3 = N_1\ , \qquad \rho x_4 = -T\ , \tag{19}$$

where ρ is a proportionality factor (see Ex. 39 of Chap. III). These equations are quite useful in comparing results of different authors.

The reader who wishes to compare some of the original memoirs of Wilczynski and his followers with their results as reported here may find the following remarks helpful. Wilczynski originally wrote the equations defining a surface projectively in ordinary space in the form

$$\begin{cases} y_{uu} + 2a y_u + 2b y_v + c y = 0\ , \\ y_{vv} + 2a' y_u + 2b' y_v + c' y = 0\ , \end{cases} \tag{20}$$

the asymptotic curves obviously being parametric. He frequently used a canonical form of these equations (see Ex. 2 of Chap. III), namely,

$$\begin{cases} y_{uu} + 2b y_v + f y = 0\ , \\ y_{vv} + 2a' y_u + g y = 0\ , \end{cases} \tag{21}$$

and at the same time employed the local tetrahedron whose vertices are the points y, y_u, y_v, y_{uv}. Only the first of these points is covariant to the surface.

It can be shown,† although we shall not reproduce the proof here, that a local equation written in Wilczynski's canonical notation and referred to the

* Fubini, 1918. 1, p. 1035.

† Lane, 1926. 10, p. 369.

corresponding local coordinate system (in which a point has coordinates $y_1, \ldots, y_4$) can be transformed into the notation of Fubini's canonical form and referred to the corresponding local coordinate system (in which a point has the coordinates $x_1, \ldots, x_4$ that have been used in this book) by proceeding as follows. *First replace f, g by the expressions*

$$f=c-a_u-a^2-2bb', \qquad g=c'-b'_v-b'^2-2aa'. \tag{22}$$

Then make the transformation

$$\begin{cases} y_1=x_1-ax_2-b'x_3+(ab'-a_v)x_4, \\ y_2=x_2-b'x_4, \qquad y_3=x_3-ax_4, \qquad y_4=x_4. \end{cases} \tag{23}$$

Finally make the substitution

$$\begin{pmatrix} a & b & c & a' & b' & c' \\ -\theta_u/2 & -\beta/2 & -p & -\gamma/2 & -\theta_v/2 & -q \end{pmatrix}. \tag{24}$$

The inverse transformation is not difficult to formulate and will be left to the reader.

At the end of Section 18 reference was made to Wilczynski's definition of his canonical quadric, which we shall now explain. Wilczynski used* in place of our expansion (III, 18) an expansion

$$\bar{z}=\bar{x}\bar{y}+(\bar{x}^3+\bar{y}^3)/6+(I\bar{x}^4+J\bar{y}^4)/24+\cdots, \tag{25}$$

where I, J are two absolute invariants which we do not need to define any more explicitly at the moment. The coordinate system is covariant. Without giving here the complete geometrical description of the coordinate system we note that the three edges through the point $P(0, 0, 0)$ under consideration are the asymptotic tangents and the directrix d_1, while the directrix d_2 in the tangent plane is still another edge. It was in completing the geometric description of the coordinate system that Wilczynski was led to introduce his canonical quadric and cubic. It can be shown, but we shall not give the details of the argument, that the transformation from this coordinate system to the coordinate system used in this book is capable of being decomposed into the following three steps, when the differential equations are in Fubini's canonical form:

$$\begin{cases} \bar{x}=-2x', & \bar{y}=-2y', & \bar{z}=4z', \\ x'=(X+AZ)/D, & y'=(Y+BZ)/D, & z'=Z/D, \\ X=lx, & Y=my, & Z=\beta\gamma z, \end{cases} \tag{26}$$

* Wilczynski, 1908. 2, p. 103.

where

$$(27) \qquad \begin{cases} D=1+BX+AY+(AB+\theta_{uv}/2\beta\gamma)Z\ , \\ A=\psi/2m\ , \quad B=\varphi/2l\ , \quad l=(\beta^2\gamma)^{1/3}\ , \quad m=(\beta\gamma^2)^{1/3}\ . \end{cases}$$

Wilczynski's definition of his canonical quadric depends on his *canonical cubic,* whose equation in his coordinate system is

$$(28) \qquad \bar{z}=\bar{x}\bar{y}+(\bar{x}^3+\bar{y}^3)/6\ .$$

By the transformation (26) this equation becomes, in our homogeneous coordinates,

$$(29) \qquad \begin{cases} [x_1x_4-x_2x_3+\theta_{uv}x_4^2/2][2x_1+\varphi x_2+\psi x_3+(\theta_{uv}+\varphi\psi/2)x_4] \\ \qquad\qquad +(2/3)[\beta(x_2+\psi x_4/2)^3+\gamma(x_3+\varphi x_4/2)^3]=0\ . \end{cases}$$

This cubic was defined by Wilczynski by means of the following properties. It has a *unode,* i.e., a point where the tangent plane is indeterminate and the polar quadric cone is two coincident planes, at the point (0, 0, 0, 1) on the directrix d_1 through the point P, such that the *uniplane,* $\bar{x}_1=0$, contains the reciprocal directrix d_2. It has third-order contact with the surface at the point P, and the four tangents of fourth-order contact form a harmonic set in which conjugate pairs are actually conjugate tangents of the surface. The reader may refer to the *Second Memoir* of Wilczynski just cited for the proofs of these properties.

In Wilczynski's coordinate system the equation of his *canonical quadric* is

$$(30) \qquad \bar{z}-\bar{x}\bar{y}=0\ .$$

By the transformation (26) this equation becomes

$$(31) \qquad 2(x_1x_4-x_2x_3)+\theta_{uv}x_4^2=0\ .$$

The essential part of Wilczynski's definition of this quadric can be formulated by saying that *this quadric is the only quadric of Darboux that is tangent to the uniplane,*

$$(32) \qquad 2x_1+\varphi x_2+\psi x_3+(\theta_{uv}+\varphi\psi/2)x_4=0\ ,$$

of the canonical cubic. This definition used by Wilczynski should be compared with the very elegant definition given nearly two decades later by Bompiani, as stated in Section 18.

Incidentally, we may note that the fixed uniplane has a variable pole with respect to a variable quadric of Darboux at a point P, and that the locus of this pole is the directrix d_1 through P. When the pole is on the plane, the plane is tangent to the canonical quadric at the point distinct from P where the directrix d_1 pierces it.

The process used by Wilczynski in truncating his canonical expansion to obtain a canonical quadric and a canonical cubic suggests the following developments. Let us consider our canonical expansion (III, 18), for which the geometric significance of the coordinate system is already known, and let us truncate this series after the terms of degree n, obtaining thus a canonical algebraic surface of degree n. This surface can be characterized geometrically by two properties.* First, *the algebraic surface has contact of order n with the analytic surface* (III, 18) *at the point P* (1, 0, 0, 0). Second, *the algebraic surface has a unode of order $n-1$ at the point* (0, 0, 0 ,1), *the uniplane being the plane $x_1=0$*; this means that the tangent plane and the polar surfaces of the algebraic surface at (0, 0, 0, 1) up to and including the one of order $n-2$ are indeterminate, while the one of order $n-1$ degenerates into the plane $x_1=0$ counted $n-1$ times.

61. Congruences in ordinary space. The purpose of this section is to outline the theory of rectilinear congruences in ordinary space as developed by Wilczynski in his prize memoir.† We shall preserve Wilczynski's notations as far as feasible, and shall present some of the fundamental geometrical results.

The defining system of differential equations can be established in the following way. It was remarked in Section 32 that a congruence in space S_n can be represented analytically by a pair of equations of the form

$$y_v=mz\ , \qquad z_u=ny\ .$$

With this representation the surfaces S_y, S_z are the focal surfaces (supposed distinct) of the congruence; these surfaces are referred to the conjugate nets $dudv=0$ in which the developables of the congruence touch them; a generator yz of the congruence touches S_y, S_z in two points P_y, P_z with the same curvilinear coordinates u, v; and the proportionality factors of y, z have been chosen in a special way. When the congruence is in ordinary space S_3 two more equations appear. The points y, z, y_u, z_v are not ordinarily coplanar, since two distinct surfaces do not ordinarily have the same tangent planes. Then the determinant (y, z, y_u, z_v) does not vanish, and it

* Lane, 1927. 10, p. 812.

† Wilczynski, 1911. 3.

follows that y_{uu} and z_{vv} can be expressed as linear combinations of y, z, y_u, z_v. The result can be stated as follows.

A congruence in ordinary space is an integral congruence of a system of equations which can be reduced to the form

$$\left\{\begin{aligned} &y_v = mz \;, \qquad z_u = ny \;, \\ &y_{uu} = ay + bz + cy_u + dz_v \;, \\ &z_{vv} = a'y + b'z + c'y_u + d'z_v \;, \end{aligned}\right. \tag{33}$$

in which the coefficients are scalar functions of u, v.

It will be left to the reader to show that any projective transform of the congruence yz is also an integral congruence of the system (33), and that the coefficients of this system satisfy eight integrability conditions obtainable from the equations

$$(y_v)_{uu} = (y_{uu})_v \;, \qquad (z_u)_{vv} = (z_{vv})_u \;.$$

Two of these conditions imply $c_v = d'_u$. Consequently there exists a function f defined, except for an additive constant, by the differential equations

$$f_u = c \;, \qquad f_v = d' \;. \tag{34}$$

The seven remaining independent *integrability conditions* can now be written in the form

$$\left\{\begin{aligned} &b = -d_v - df_v \;, \qquad a' = -c'_u - c'f_u \;, \qquad mn - c'd = f_{uv} \;, \\ &m_{uu} + d_{vv} + df_{vv} + d_v f_v - f_u m_u = ma + db' \;, \\ &n_{vv} + c'_{uu} + c'f_{uu} + c'_u f_u - f_v n_v = c'a + nb' \;, \\ &2m_u n + mn_u = a_v + f_u mn + a'd \;, \\ &m_v n + 2mn_v = b'_u + f_v mn + bc' \;. \end{aligned}\right. \tag{35}$$

The most general transformation leaving the form of system (33) invariant is

$$y = \lambda\bar{y} \;, \qquad z = \mu\bar{z} \;, \qquad \bar{u} = \alpha \;, \qquad \bar{v} = \beta \qquad (\lambda\mu\alpha_u\beta_v \neq 0) \;, \tag{36}$$

where λ, α are functions of u alone, and μ, β of v alone. The effect of this transformation on system (33) is to produce another system of the same form whose coefficients, indicated by dashes, are given by the following formulas:

$$
(37) \quad \begin{cases}
\bar{m} = m\mu/\lambda\beta_v \, , & \bar{n} = n\lambda/\mu\alpha_u \, , \\
\bar{a} = (1/\alpha_u^2)(a + c\lambda_u/\lambda - \lambda_{uu}/\lambda) \, , & \bar{b}' = (1/\beta_v^2)(b' + d'\mu_v/\mu - \mu_{vv}/\mu) \, , \\
\bar{b} = (\mu/\lambda\alpha_u^2)(b + d\mu_v/\mu) \, , & \bar{a}' = (\lambda/\mu\beta_v^2)(a' + c'\lambda_u/\lambda) \, , \\
\bar{c} = (1/\alpha_u)(c - 2\lambda_u/\lambda - \alpha_{uu}/\alpha_u) \, , & \bar{d}' = (1/\beta_v)(d' - 2\mu_v/\mu - \beta_{vv}/\beta_v) \, , \\
\bar{d} = d\mu\beta_v/\lambda\alpha_u^2 \, , & \bar{c}' = c'\lambda\alpha_u/\mu\beta_v^2 \, .
\end{cases}
$$

It is possible to find a pair of equations of the form described in Exercise 13 of Chapter IV for the surface S_y, and similarly for S_z. In fact, elimination of z from system (33) and some of the equations obtainable therefrom by differentiation leads at once to the equations for S_y,

$$
(38) \quad \begin{cases}
y_{uv} = mny + (\log m)_u y_v \, , \\
y_{uu} = ay + f_u y_u + [b - d(\log m)_v] y_v/m + dy_{vv}/m \, .
\end{cases}
$$

The equations for the surface S_z can be written, whenever they are needed, by means of the following substitution:

$$
(39) \quad \begin{pmatrix} y & u & m & a & b & c & d \\ z & v & n & b' & a' & d' & c' \end{pmatrix} .
$$

Of course, in dividing by m in equations (38) we assume $m \neq 0$. The first equation of system (33) shows that *if S_y is a proper surface, then* $m \neq 0$, and similarly the second of (33) shows that *if S_z is a proper surface, then $n \neq 0$.* Direct calculation shows that the differential equations of the asymptotic curves on S_y, S_z are respectively

$$
(40) \quad d\,du^2 + m\,dv^2 = 0 \, , \qquad n\,du^2 + c'dv^2 = 0 \, .
$$

It follows that *if S_y is not a developable surface then* $d \neq 0$, and *if S_z is not developable then* $c' \neq 0$. We shall assume hereinafter that $mnc'd \neq 0$.

Equations (40) show that the asymptotic curves on S_y and S_z correspond if, and only if,

$$
mn - c'd = 0 \, .
$$

Let us call the left member of this equation the Weingarten invariant W, so that we have

$$
(41) \quad W = mn - c'd = f_{uv} \, .
$$

Then we have the theorem:

The congruence yz is a W congruence if, and only if, $f_{uv}=0$.

Let us define ρ, σ by placing

$$\rho=y_u-(\log m)_u y\ ,\qquad \sigma=z_v-(\log n)_v z\ . \tag{42}$$

Then ρ is the first Laplace transform of y in the u-direction, and σ is that of z in the v-direction. A system of equations of the form (33) for the congruence of lines ρy can be calculated. For this purpose let us place

$$Y=\rho\ ,\qquad Z=y/m\ .$$

Calculating the various derivatives of Y, Z, and eliminating y, z and their derivatives, we obtain the desired system of equations, whose coefficients, denoted by capital letters, are given by the following formulas:

$$\left\{\begin{aligned}
&M=m[mn-(\log m)_{uv}]\ ,\qquad N=1/m\ ,\\
&A=a+(f-2\log m)_{uu}+(\log d/m)_u(\log m-f)_u\ ,\\
&B=m[a_u+bn+af_u+dn_v-(\log m)_{uuu}]\\
&\qquad+(\log m)_v[b_u+bf_u+mnd-b(\log m)_u]\\
&\qquad+m_u[f_{uu}+f_u^2-f_u(\log m)_u-2(\log m)_{uu}]\\
&\qquad+(f+\log d/m)_u[m_{uu}-f_u m_u-am-b(\log m)_v]\ ,\\
&C=(f+\log d/m)_u\ ,\qquad D'=(f+\log d/m)_v\ ,\\
&D=d[c'd-(\log d)_{uv}]\ ,\qquad C'=1/d\ ,\\
&A'=(\log m-f)_u/d\ ,\\
&B'=-(\log m)_{vv}+[m_{uu}-f_u m_u-am-b(\log m)_v]/d\ .
\end{aligned}\right. \tag{43}$$

We do not need to write here the corresponding formulas for the congruence $z\sigma$, since they can be written at once by aid of the substitution (39). *The congruences $y\rho$ and $z\sigma$ are called the first and minus-first Laplace transforms of the congruence yz.*

In order to study the differential geometry of a congruence in the neighborhood of one of its generators yz it is convenient to introduce line coordinates. Let us use the tetrahedron y, z, ρ, σ as a local tetrahedron of reference with the convention that a point $x_1y+x_2z+x_3\rho+x_4\sigma$ shall have local coordinates proportional to $x_1, \ldots, x_4$. Then for the local coordinates $y_1, \ldots, y_4$ of a point Y near the point P_y on the surface S_y we find by the usual method the following power series expansions:

$$
(44)\qquad
\begin{cases}
y_1 = 1 + (\log m)_u \Delta u + [a + f_u(\log m)_u]\Delta u^2/2 + mn\Delta u\Delta v + \cdots,\\
y_2 = m\Delta v + [b + d(\log n)_v]\Delta u^2/2 + m_u\Delta u\Delta v + m(\log mn)_v\Delta v^2/2 + \cdots,\\
y_3 = \Delta u + f_u\Delta u^2/2 + \cdots,\\
y_4 = d\Delta u^2/2 + m\Delta v^2/2 + \cdots.
\end{cases}
$$

The corresponding expansions for the coordinates of the point Z near the point P_z on the surface S_z can be found in the same way, or else can be written by means of the substitution (39) augmented by

$$
(45)\qquad
\begin{pmatrix} 1 & 3 & \rho \\ 2 & 4 & \sigma \end{pmatrix}.
$$

These expansions are

$$
(46)\qquad
\begin{cases}
z_1 = n\Delta u + n(\log mn)_u\Delta u^2/2 + n_v\Delta u\Delta v + [a' + c'(\log m)_u]\Delta v^2/2 + \cdots,\\
z_2 = 1 + (\log n)_v\Delta v + mn\Delta u\Delta v + [b' + d'(\log n)_v]\Delta v^2/2 + \cdots,\\
z_3 = n\Delta u^2/2 + c'\Delta v^2/2 + \cdots,\\
z_4 = \Delta v + f_v\Delta v^2/2 + \cdots.
\end{cases}
$$

The local coordinates ω_{ik} of the generator YZ near the generator yz of the congruence are defined by the familiar formulas (I, 42) and are found to be represented by the series

$$
(47)\qquad
\begin{cases}
\omega_{12} = 1 + (\log m)_u\Delta u + (\log n)_v\Delta v + [a + f_u(\log m)_u]\Delta u^2/2\\
\qquad\qquad + [mn + (\log m)_u(\log n)_v]\Delta u\Delta v + [b' + f_v(\log n)_v]\Delta v^2/2 + \cdots,\\
\omega_{13} = -n\Delta u^2/2 + c'\Delta v^2/2 + \cdots,\\
\omega_{14} = \Delta v + (\log m)_u\Delta u\Delta v + f_v\Delta v^2/2 + \cdots,\\
\omega_{23} = -\Delta u - f_u\Delta u^2/2 - (\log n)_v\Delta u\Delta v + \cdots,\\
\omega_{42} = d\Delta u^2/2 - m\Delta v^2/2 + \cdots,\\
\omega_{34} = \Delta u\Delta v + \cdots.
\end{cases}
$$

We observe that the complex $\omega_{13}=0$ is special and consists of all lines intersecting the line $z\sigma$. Similarly, the complex $\omega_{42}=0$ consists of all lines intersecting the line $y\rho$; and the complex $\omega_{34}=0$ consists of all lines intersecting the generator yz. The lines common to all three of these complexes form two flat pencils, one with its center at the point P_y and lying in the tangent plane of the surface S_z at the point P_z, and the other with its center at P_z and lying in the tangent plane of S_y at P_y. *The lines of these two pencils are called the central rays of the generator* yz.

The central rays (see Fig. 42) can also be characterized in another way. If we write the equation (I, 44) of a general linear complex and demand that it be satisfied by the series (47) for ω_{ik} identically in Δu, Δv as far as the

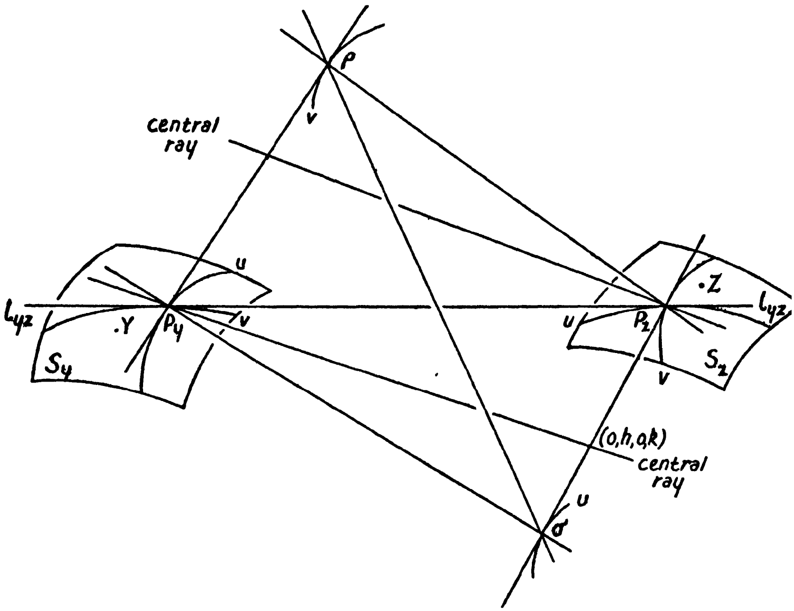

FIG. 42

terms of the first degree, we obtain the conditions $a_{34}=a_{23}=a_{14}=0$. Thus the equation of the most general linear complex having first-order contact with the congruence yz along a generator l_{yz} is found to be

$$a_{42}\omega_{13}+a_{13}\omega_{42}+a_{12}\omega_{34}=0 \ . \tag{48}$$

Such a complex contains the generator l_{yz} and a consecutive generator of every ruled surface in the congruence and containing l_{yz}. There are obviously ∞^2 such linear complexes; all of them have in common the central rays of the generator l_{yz} and no other lines.

If we go on and seek to determine a linear complex having second-order contact with the congruence yz along a generator l_{yz}, we find that such a complex exists if, and only if, the congruence is a W congruence. In this case its equation is

$$d\omega_{13}+n\omega_{42}=0 \ , \tag{49}$$

and it is called *the osculating linear complex* along the generator l_{yz} of the W congruence yz.

There is a unique linear complex associated* with one focal point P_y of a generator l_{yz} of a congruence yz in the following manner. The complex contains l_{yz} and all its central rays through P_y. Moreover, for every ruled surface of the congruence through l_{yz}, the complex contains also the generator YZ consecutive to l_{yz} and all its central rays through its focal point Y consecutive to P_y. This complex is called *the associated linear complex* of the focal point P_y of the generator l_{yz} of the congruence. There is, of course, a complex similarly associated with the other focal point P_z. In order to find the equation of the associated linear complex of the point P_y we proceed as follows. A linear complex containing the line l_{yz} $(1, 0, 0, 0, 0, 0)$ must have $a_{34}=0$. A central ray through the point P_y $(1, 0, 0, 0)$ and an arbitrary point $(0, h, 0, k)$ on the line $z\sigma$ has coordinates $(h, 0, k, 0, 0, 0)$, and belongs to the complex in case also $a_{23}=0$. The series (47) show that the complex contains an arbitrary generator YZ consecutive to l_{yz} in case also $a_{14}=0$. The equation of the complex now has the form (48). We proceed to calculate the coordinates of an arbitrary central ray of a neighboring generator YZ through the point Y. The local coordinates of the point Z are given by equations (46). The local coordinates of the corresponding point Σ near the point σ are found in the usual way to be given by

$$(50)\qquad \begin{cases} \sigma_1=[a'+c'(\log m)_u]\Delta v+\cdots, \\ \sigma_2=[mn-(\log n)_{uv}]\Delta u+[b'+f_v(\log n)_v-n_{vv}/n]\Delta v+\cdots, \\ \sigma_3=c'\Delta v+\cdots, \\ \sigma_4=1+(f-\log n)_v\Delta v+\cdots. \end{cases}$$

The local coordinates of a point $hZ+k\Sigma$ can easily be written, and then the needed line coordinates of the central ray joining the point Y to the point $hZ+k\Sigma$ are found to be

$$(*,\ kc'\Delta v+\cdots,\ *,\ *,\ -km\Delta v+\cdots,\ k\Delta u+\cdots),$$

the unwritten coordinates being not needed for our purpose. These coordinates satisfy equation (48) identically in h, k and to terms of the first degree in Δu, Δv in case

$$a_{12}=0\ ,\qquad a_{42}c'-a_{13}m=0\ .$$

* Waelsch, *Zur Infinitesimalgeometrie der Strahlencongruenzen und Flächen*, "Sitzungsberichte der Wiener Akademie der Wissenschaften," C, Abt. II*a* (1891), 167.

Thus the local equation of *the associated complex of the point P_y* of the generator yz is found to be

$$m\omega_{13}+c'\omega_{42}=0 \ . \tag{51}$$

Similarly, the equation of *the associated complex of P_z* is

$$d\omega_{13}+n\omega_{42}=0 \ . \tag{52}$$

These complexes are the same and are the osculating linear complex if, and only if, the congruence is a W congruence.

If the congruence is not a W congruence the associated complexes (51), (52) are distinct and determine a pencil of linear complexes, whose equation can be written in the form

$$h(m\omega_{13}+c'\omega_{42})+k(d\omega_{13}+n\omega_{42})=0 \ .$$

The special complexes of this pencil are given by

$$(hm+kd)(hc'+kn)=0 \ ,$$

and consequently their equations are $\omega_{42}=0$ and $\omega_{13}=0$ respectively. The first of these, as we have already observed, consists of all lines intersecting the line $y\rho$, and the second bears the same relation to the line $z\sigma$. Therefore *the two lines $y\rho$, $z\sigma$ are the directrices of the congruence of intersection of the associated complexes* (51), (52). These two lines are sometimes called *the principal focal rays* of the generator yz, all of the focal rays being the tangents of the surface S_y at the point P_y and the tangents of S_z at P_z.

EXERCISES

1. Use equations (I, 45) to prove that the planes corresponding to any point of space in the null systems of four linear complexes of a pencil themselves belong to an axial pencil. Prove that the cross ratio of the four planes is independent of the point used, so that it may properly be called the cross ratio of the four complexes. Prove that the cross ratio of the complexes (51), (52) and $\omega_{42}=0$, $\omega_{13}=0$ is $mn/c'd$.

2. Calculate the system of four differential equations expressing the four third derivatives of the line coordinates ω of a generator of a congruence, not a W congruence, as linear homogeneous combinations of ω, ω_u, ω_v, ω_{uu}, ω_{uv}, ω_{vv}. Discuss the situation for a W congruence. Wilczynski, 1911. 3, p. 30

3. If a congruence belongs to a linear complex, and if the first Laplace transformed congruence also belongs to a linear complex, then all the Laplace trans-

formed congruences belong to linear complexes. The second, fourth, sixth, etc. transformed congruences are projectively equivalent to the original congruence; the third, fifth, etc. are projectively equivalent to the first Laplace transformed congruence. WILCZYNSKI, 1911. 3, p. 69

4. The following formulas give invariants of the parametric conjugate nets on the surfaces S_y (left column) and S_z (right column) of the last section:

$$
\begin{array}{ll}
H_y = mn\,, & H_z = mn - (\log n)_{uv}\,, \\
K_y = mn - (\log m)_{uv}\,, & K_z = mn\,, \\
W_y^{(u)} = -(f + \log d/m)_{uv}\,, & W_z^{(u)} = -f_{uv}\,, \\
W_y^{(v)} = -f_{uv}\,, & W_z^{(v)} = -(f + \log c'/n)_{uv}\,, \\
\mathfrak{D}_y = a + f_u(\log m)_u - m_{uu}/m\,, & \mathfrak{D}_z = -(n/c')[b' + f_v(\log n)_v - n_{vv}/n]\,, \\
\mathscr{H}_y = mn - (f + \log d)_{uv}\,, & \mathscr{H}_z = mn - f_{uv}\,, \\
\mathscr{K}_y = mn - f_{uv}\,, & \mathscr{K}_z = mn - (f + \log c')_{uv}\,, \\
8\mathfrak{B}_y' = -(2f + \log d^3 m)_v\,, & 8\mathfrak{B}_z' = -(2f - \log c' n^3)_v\,, \\
8\mathfrak{C}_y' = -(2f - \log d m^3)_u\,, & 8\mathfrak{C}_z' = -(2f + \log c'^3 n)_u\,.
\end{array}
$$

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ABBREVIATIONS

The following list of abbreviations, which will be used hereinafter, is extracted from the list on pages 13–17 of *Selected Topics in Algebraic Geometry*, Bulletin 63 of the National Research Council.

1. *Am. J.: American Journal of Mathematics*
2. *Am. M.S. Bull.: American Mathematical Society Bulletin*
3. *Am. M.S. Trans.: American Mathematical Society Transactions*
4. *Ann. di mat.: Annali di matematica pura ed applicata*
5. *Ann. of Math.: Annals of Mathematics*
6. *Berlin Sitzungsber.: Sitzungsberichte der Berliner Mathematischen Gesellschaft*
7. *Boll. d.U.M. It.: Bollettino della Unione Matematica Italiana*
8. *Bologna Mem.:* Bologna. *Reale Accademia delle Scienze dell'Istituto Memorie*
9. *Brux. Bull.:* Bruxelles. Académie Royale de Belgique. *Bulletins*

10. *Bull. sci. m.: Bulletin des sciences mathématiques*
11. *C.R.: Comptes Rendus.* Académie des Sciences de l'Institut de France. Paris
12. *J. de math.: Journal de mathématiques pures et appliquées. Liouville*
13. *J. Ec. Polyt.: Journal de l'Ecole Polytechnique.* Paris
14. *Lomb. Ist. Rend.:* Reale Istituto Lombardo di Scienze e Lettere. Milano. *Rendiconti*
15. *London M.S. Proc.: London Mathematical Society Proceedings*
16. *Math. Ann.: Mathematische Annalen*
17. *Math. Zschr.: Mathematische Zeitschrift.* Berlin
18. *Nap. R.A. Rend.:* Napoli. Reale Accademia delle Scienze Fisiche et Matematiche. *Rendiconti*
19. *Pal. Circ. Mat.:* Palermo. Circolo Matematico. *Rendiconti*
20. *Rom. Acc. L. Rend.:* Roma. Reale Accademia dei Lincei. *Rendiconti*
21. *Tôhoku M.J.: Tôhoku Mathematical Journal*
22. *Torino Atti:* Torino. Reale Accademia delle Scienze. *Atti*
23. *Torino Mem.:* Torino. Reale Accademia delle Scienze. *Memorie*
24. *Wash. Nat. Ac. Sc. Proc.:* Washington. *National Academy of Sciences Proceedings*

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